Mathematical Induction

Part Two

Outline for Today

- ***Variations on Induction***
    - Starting later, taking different step sizes, and more!
- ***“Build Up” versus “Build Down”***
    - An inductive nuance that follows from our general proofwriting principles.
- ***Complete Induction***
    - When one assumption isn’t enough!

Recap from Last Time

Let P be some predicate. The ***principle of mathematical induction*** states that if

If it starts true…

$P(0)$ is true

…and it stays true…

and

$$\forall k \in \mathbb{N}.\ (P(k) \rightarrow P(k+1))$$

then

$$\forall n \in \mathbb{N}.\ P(n)$$

…then it's always true.

Theorem: The sum of the first n powers of two is $2^n - 1$.

Proof: Let $P(n)$ be the statement "the sum of the first n powers of two is $2^n - 1$." We will prove, by induction, that $P(n)$ is true for all $n \in \mathbb{N}$, from which the theorem follows.

For our base case, we need to show $P(0)$ is true, meaning that the sum of the first zero powers of two is $2^0 - 1$. Since the sum of the first zero powers of two is zero and $2^0 - 1$ is zero as well, we see that $P(0)$ is true.

For the inductive step, assume that for some arbitrary $k \in \mathbb{N}$ that $P(k)$ holds, meaning that

$$2^0 + 2^1 + \ldots + 2^{k-1} = 2^k - 1. \qquad (1)$$

We need to show that $P(k + 1)$ holds, meaning that the sum of the first $k + 1$ powers of two is $2^{k+1} - 1$. To see this, notice that

$$\begin{aligned} 2^0 + 2^1 + \ldots + 2^{k-1} + 2^k &= (2^0 + 2^1 + \ldots + 2^{k-1}) + 2^k \\ &= 2^k - 1 + 2^k \qquad \textit{(via (1))} \\ &= 2(2^k) - 1 \\ &= 2^{k+1} - 1. \end{aligned}$$

Therefore, $P(k + 1)$ is true, completing the induction. ■

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New Stuff!

Part 2: ***How Not to Induct***

All Horses are the Same Color

$P(n)$ = “All groups of n horses always have the same color”

All Horses are the Same Color

$P(0)$ = "All groups of 0 horses always have the same color"

Vacuously true!

Base case: $n = 0$

All Horses are the Same Color

Assume $P(k)$ = "All groups of k horses always have the same color"

Inductive hypothesis: $n = k$

All Horses are the Same Color

Prove $P(k+1)$ = “All groups of $k+1$ horses always have the same color”

Inductive hypothesis: $n = k+1$

All Horses are the Same Color

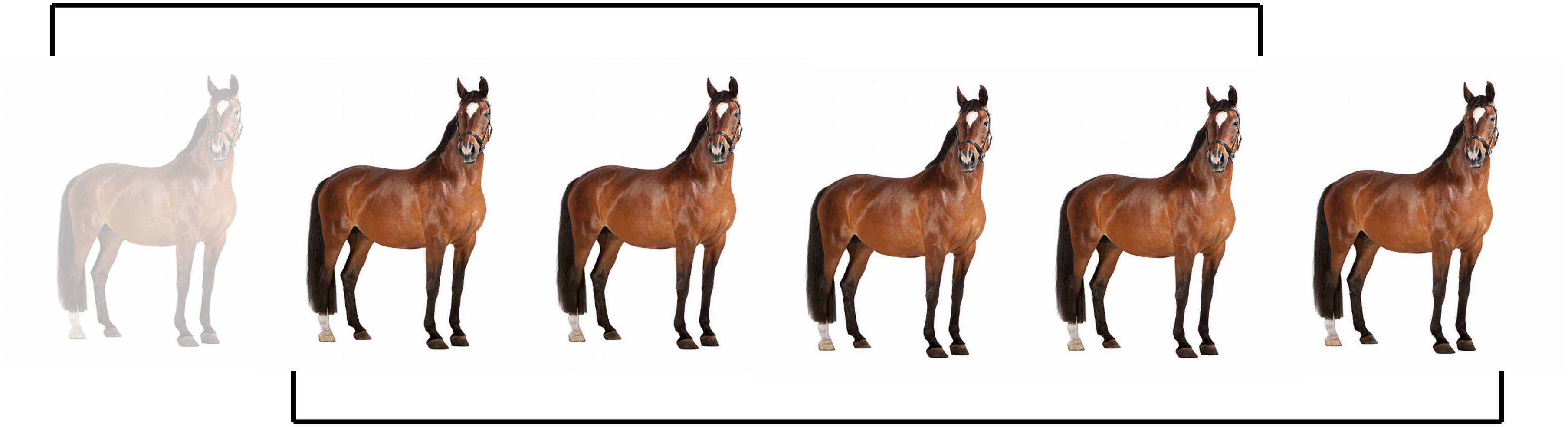
Prove $P(k+1)$ = “All groups of $k+1$ horses always have the same color”

By $P(k)$, these k horses have the same color

Inductive hypothesis: $n = k+1$

All Horses are the Same Color

Prove $P(k+1)$ = "All groups of $k+1$ horses always have the same color"



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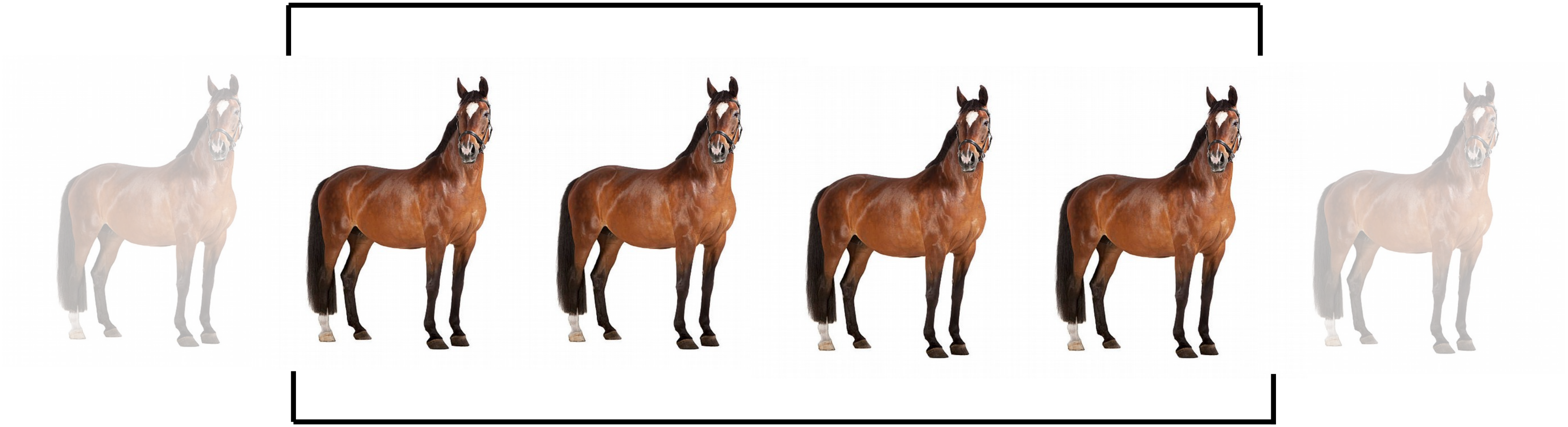
All Horses are the Same Color

Prove $P(k+1)$ = "All groups of $k+1$ horses always have the same color"

These horses in the middle were in both sets



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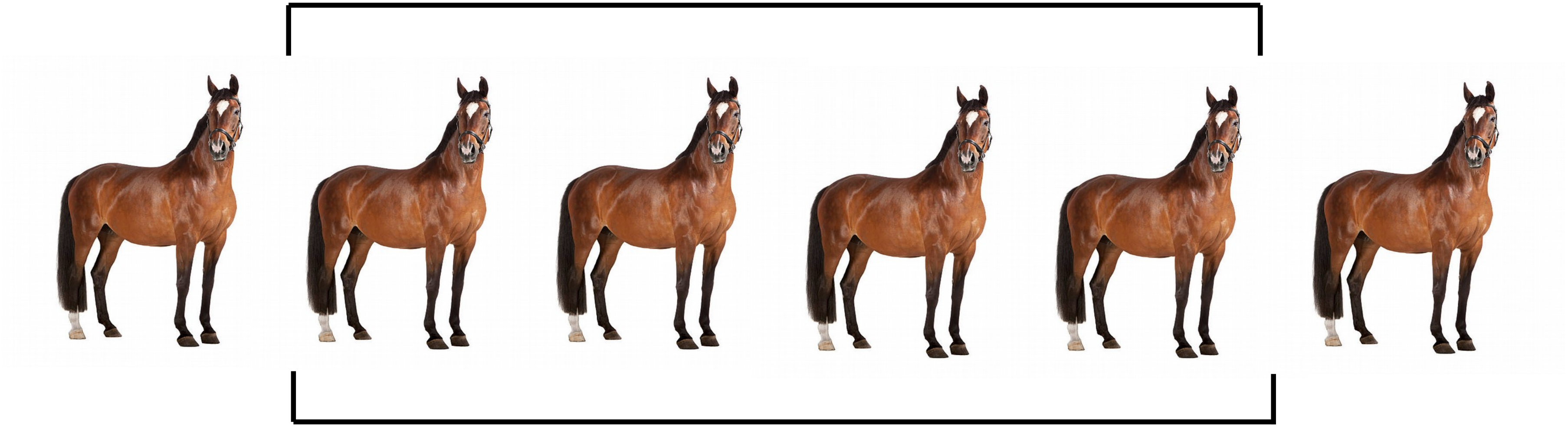


And we said that both horses on the ends are the same color as these overlapping horses

Inductive hypothesis: $n = k+1$

All Horses are the Same Color

Prove $P(k+1)$ = “All groups of $k+1$ horses always have the same color”

So all $k+1$ horses have the same color!

Inductive hypothesis: $n = k+1$

⚠ ***Incorrect!*** ⚠ ***Proof:*** Let $P(n)$ be the statement “all groups of n horses are the same color.” We will prove by induction that $P(n)$ holds for all natural numbers n, from which the theorem follows.

As our base case, we prove $P(0)$, that all groups of 0 horses are the same color. This statement is vacuously true because there are no horses.

For the inductive step, assume that for an arbitrary natural number k that $P(k)$ is true and that all groups of k horses are the same color. Now consider a group of $k+1$ horses. Exclude the last horse and look only at the first k horses. By the inductive hypothesis, these horses are the same color. Next, exclude the first horse and look only at the last k horses. Again we see by the inductive hypothesis that these horses are the same color.

Therefore, the first horse is the same color as the non-excluded horses, who in turn are the same color as the last horse. Hence the first horse excluded, the non-excluded horses, and last horse excluded are all of the same color. Thus $P(k+1)$ holds,

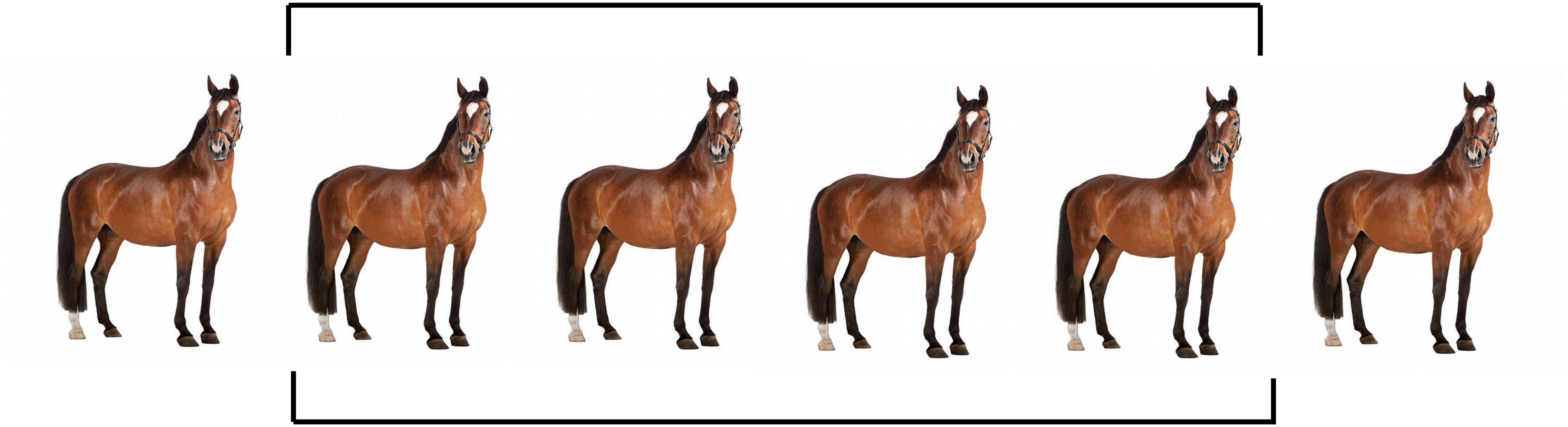
What’s wrong with this proof?
(again by paragraph number)

What’s going on here?

All Horses are the Same Color

Prove $P(k+1)$ = “All groups of $k+1$ horses always have the same color”

These horses in the middle were in both sets

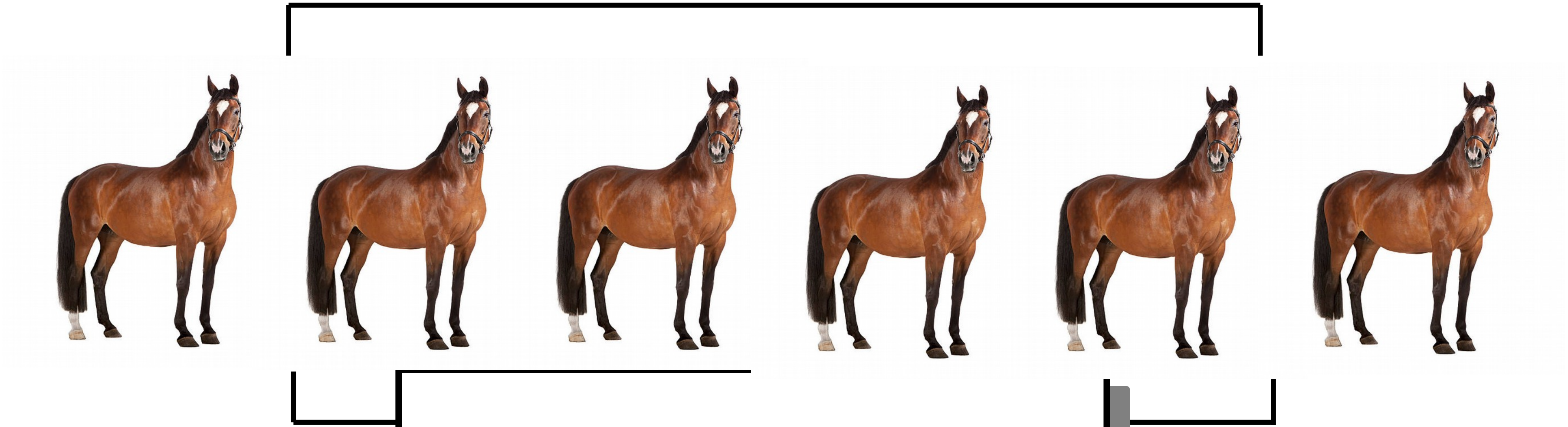


Inductive hypothesis: $n = k+1$

All Horses are the Same Color

Prove $P(k+1)$ = “All groups of $k+1$ horses always have the same color”

These horses in the middle were in both sets



But what if there are no such horses?

Inductive hypothesis: $n = k+1$

All Horses are the Same Color

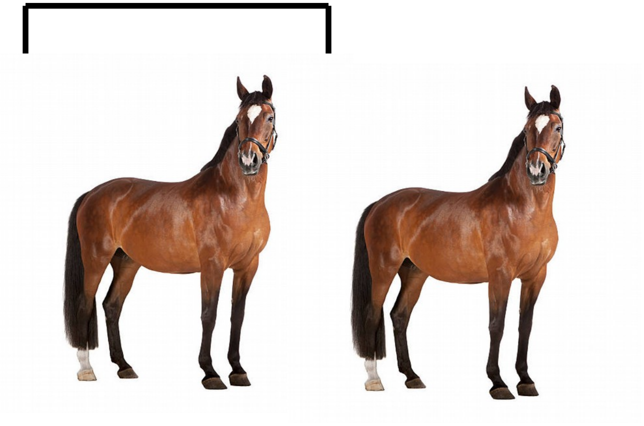
$P(n)$ = “All groups of n horses always have the same color”

$$P(1) \rightarrow P(2)$$

All Horses are the Same Color

$P(n)$ = “All groups of n horses always have the same color”

By $P(1)$, this 1 horse has the same color



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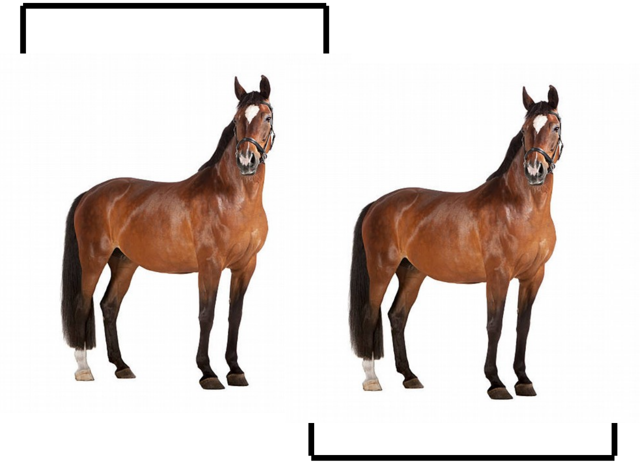

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All Horses are the Same Color

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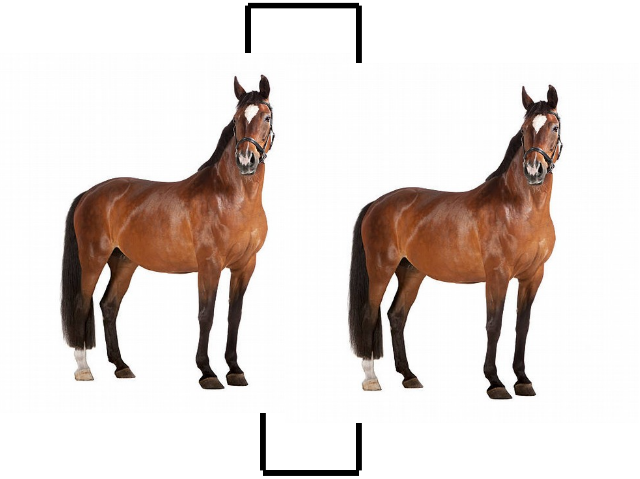
These horses in the middle (??) were in both sets



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For the inductiv al number k
that $P(k)$ is tru me color. Now
consider a grou d look only at
the first k hors es are the
same color. Nex the last k
horses. Again w se horses are
the same color.

The logic in our inductive step does not allow us to get from $P(1)$ to $P(2)$.
Specifically, there are no non-excluded horses that were in both sets.

Therefore, the first horse is the same color as the non-excluded horses, who in turn are the same color as the last horse. Hence the first horse excluded, the non-excluded horses, and last horse excluded are all of the same color. Thus $P(k+1)$ holds, completing the induction. ■

Non-Issues with this Proof

- *“We should have proven additional base cases”*
  - A proof by induction only needs a single base case, so the fact that we only have one here is not in itself an issue.
- *“We should have used complete induction”*
  - Complete induction wouldn’t have helped us here either, since our inductive step would still need to use $P(0)$ and $P(1)$ to prove $P(2)$.

Induction Debugging Tips

- Remember that induction requires two parts: the base case and the inductive step
- If you see an induction proof of a false statement, one of these pieces must be broken
- Recommendation: try playing the induction out one step at a time (Is the base case true? From the base case, does the reasoning in your inductive step allow you to conclude the next statement? What about the following statement? The one after that? etc.)

Variations on Induction: ***Starting Later***

Induction Starting at 0

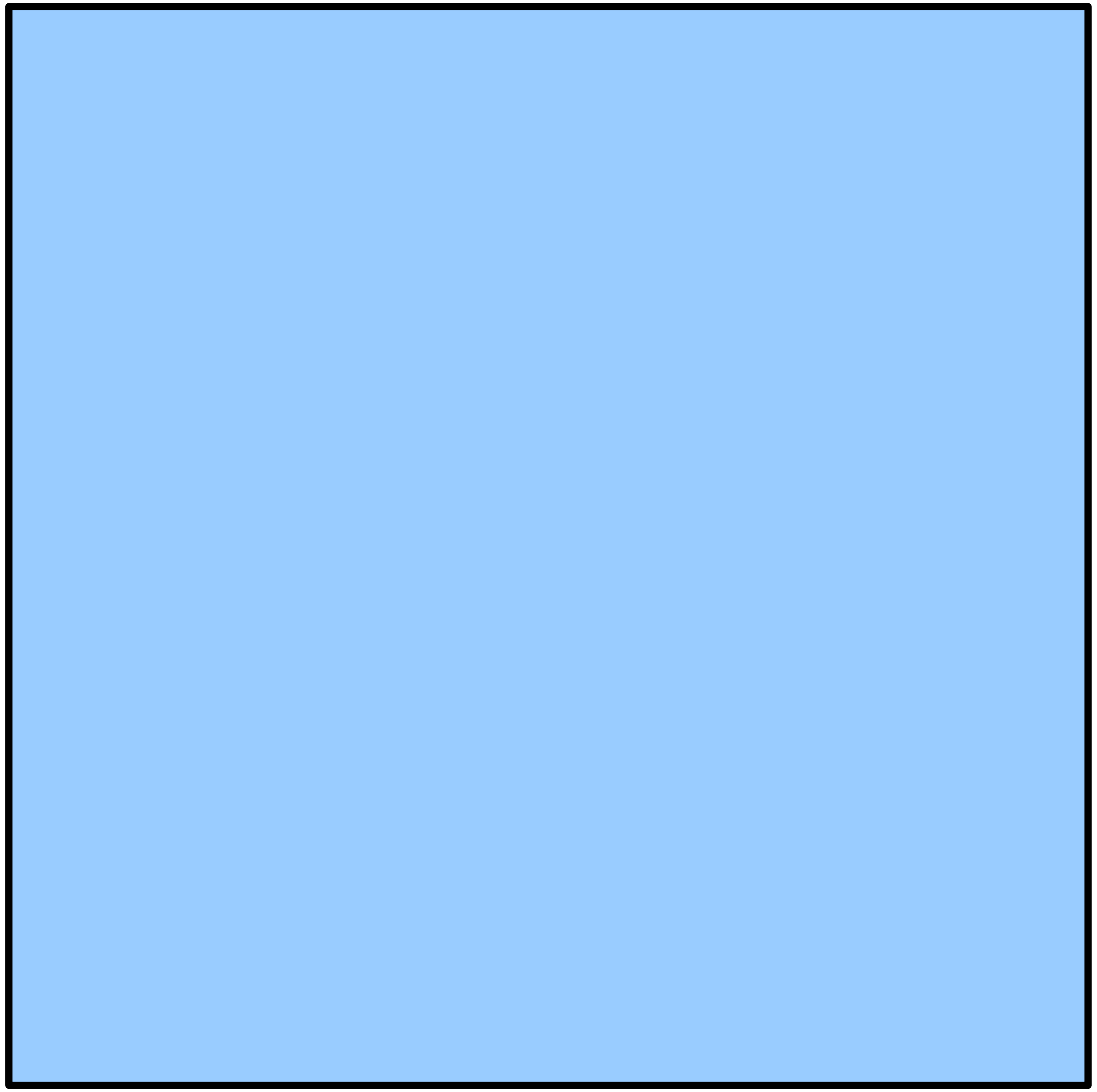
- To prove that $P(n)$ is true for all natural numbers greater than or equal to 0:
  - Show that $P(0)$ is true.
  - Show that for any $k \geq 0$, that
    if $P(k)$ is true, then $P(k+1)$ is true.
  - Conclude $P(n)$ holds for all natural numbers greater than or equal to 0.

Induction Starting at $\boldsymbol{m}$

- To prove that $P(n)$ is true for all natural numbers greater than or equal to $\boldsymbol{m}$:
  - Show that $P(\boldsymbol{m})$ is true.
  - Show that for any $k \geq \boldsymbol{m}$, that if $P(k)$ is true, then $P(k+1)$ is true.
  - Conclude $P(n)$ holds for all natural numbers greater than or equal to $\boldsymbol{m}$.

Variations on Induction: ***Bigger Steps***

Subdividing a Square



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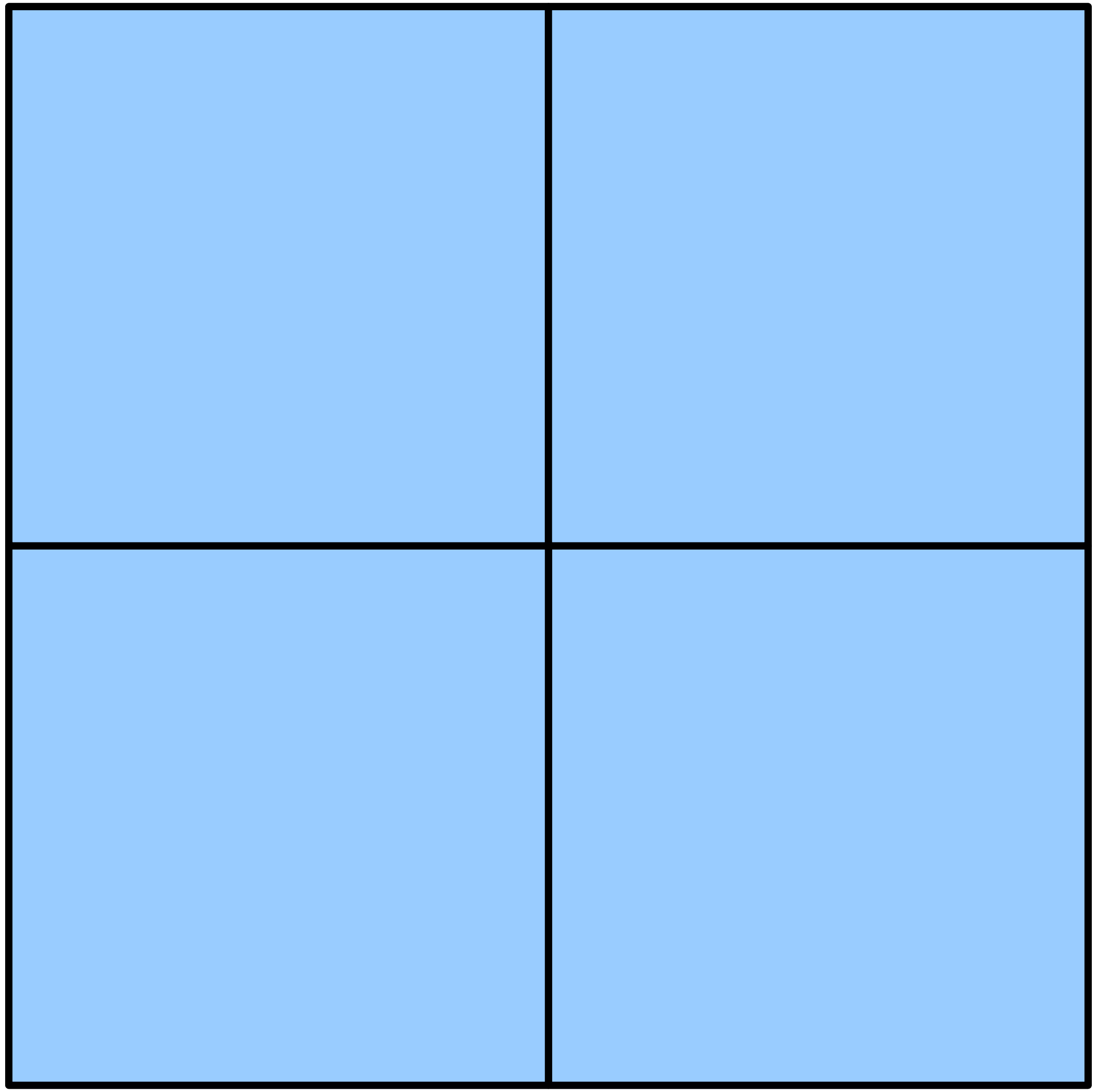

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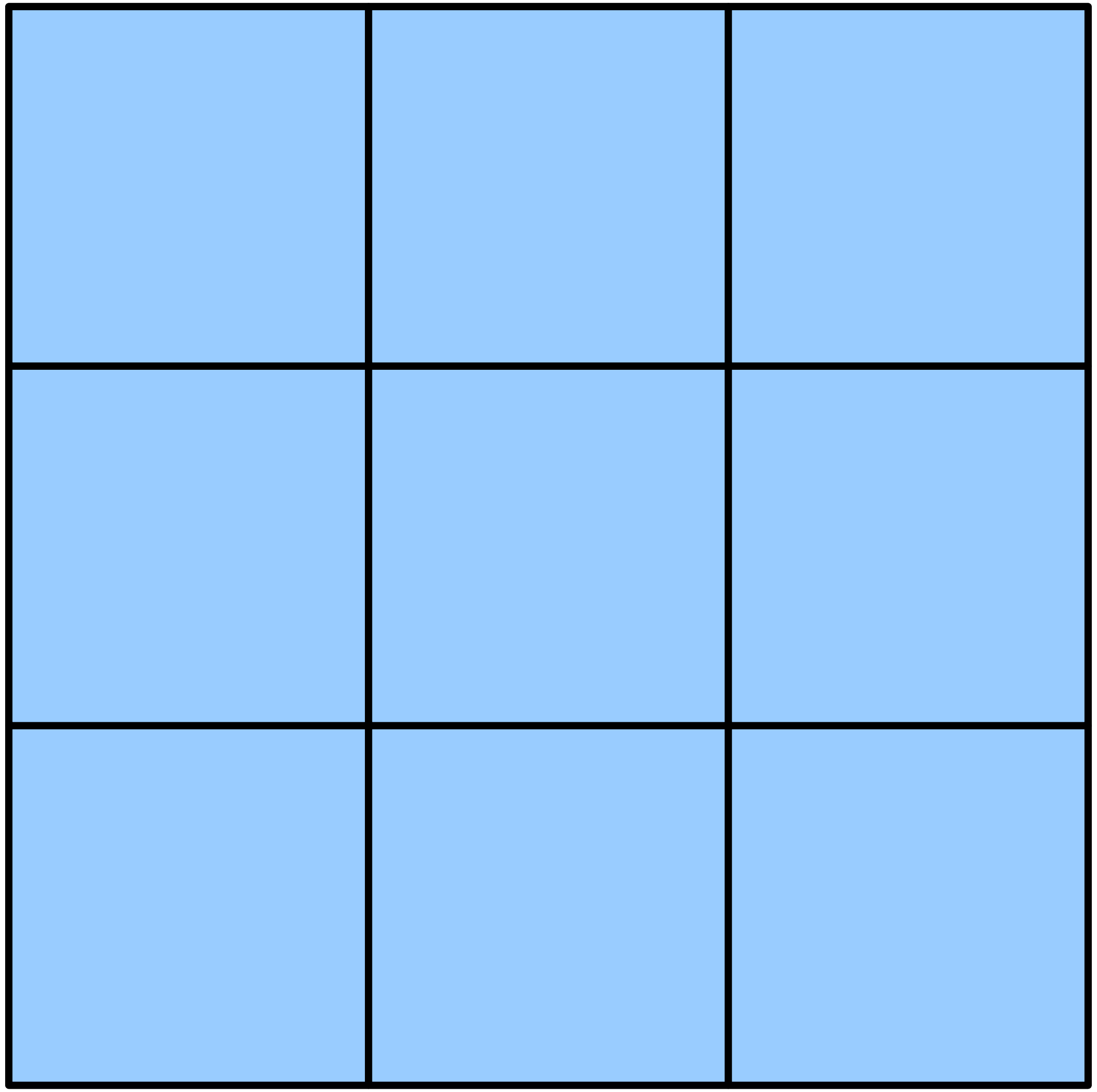

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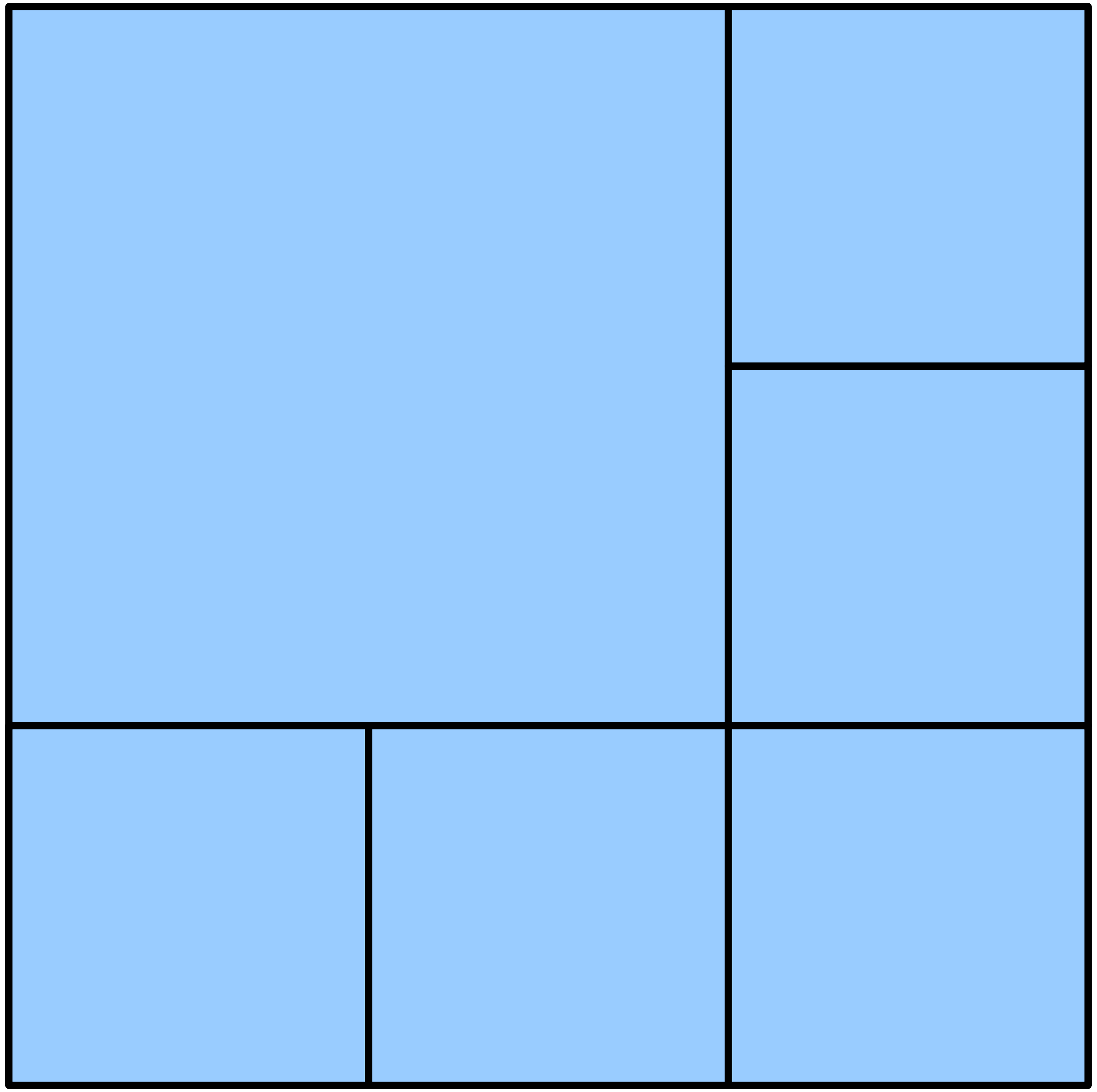
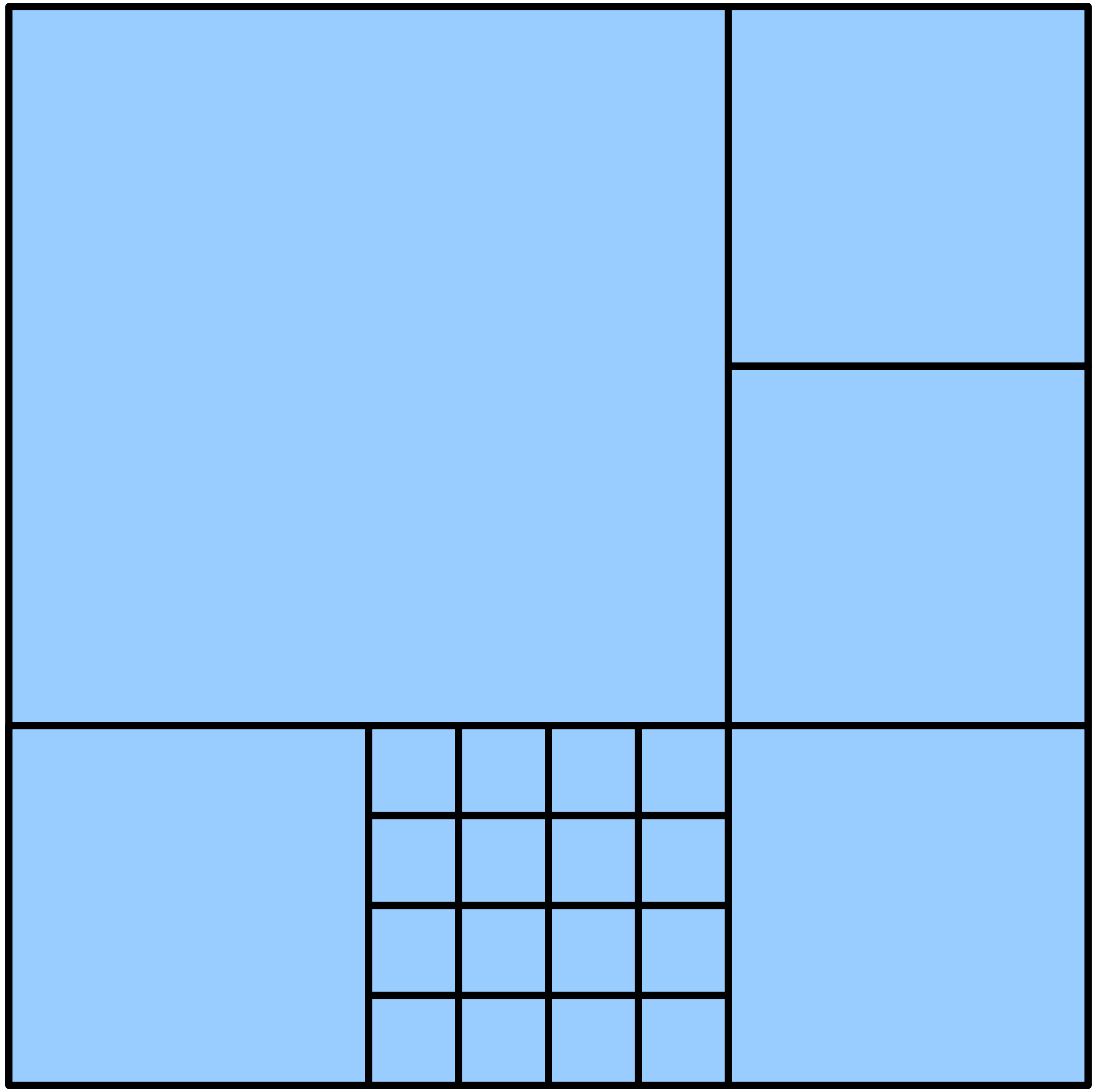


Subdividing a Square



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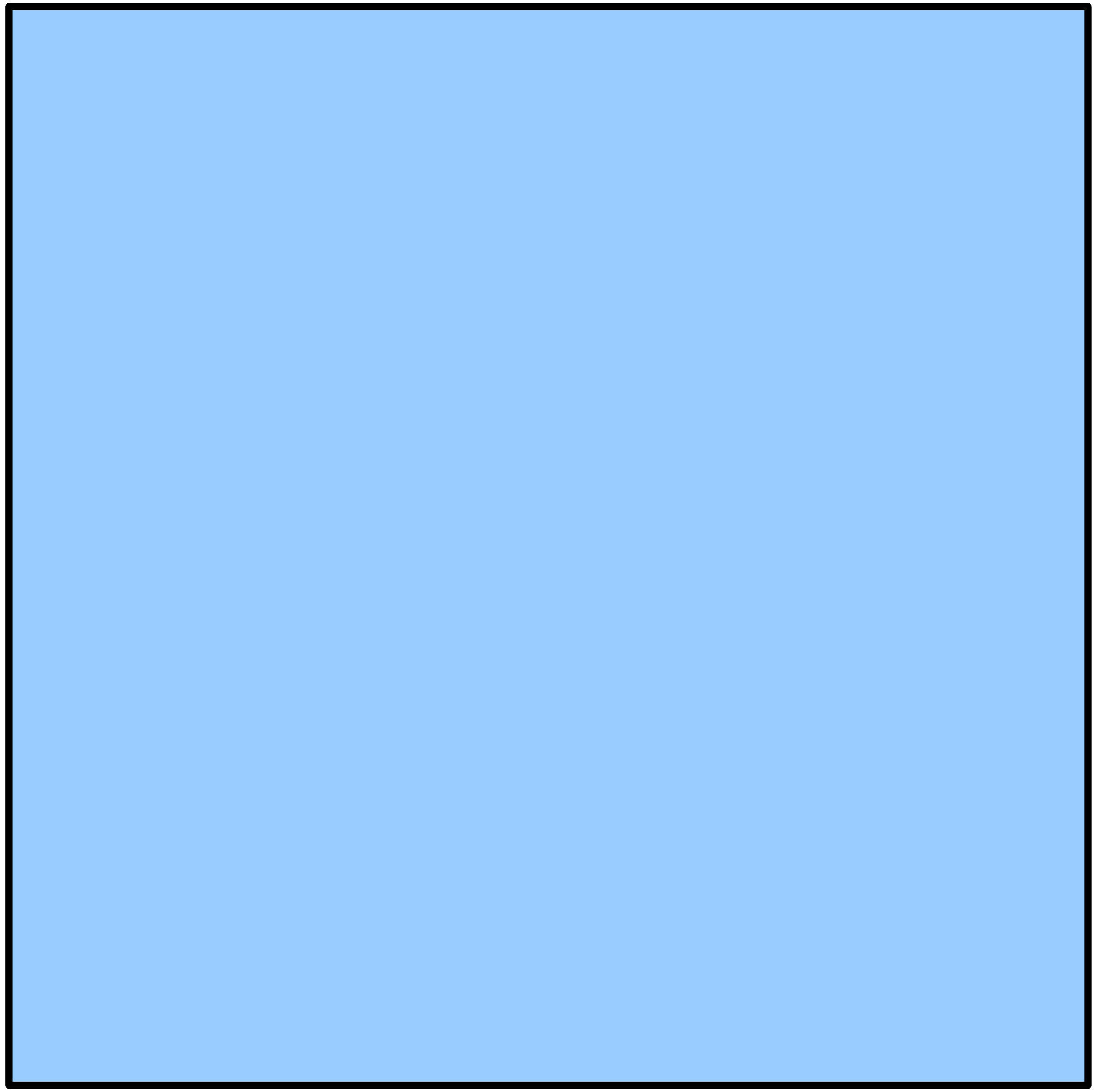

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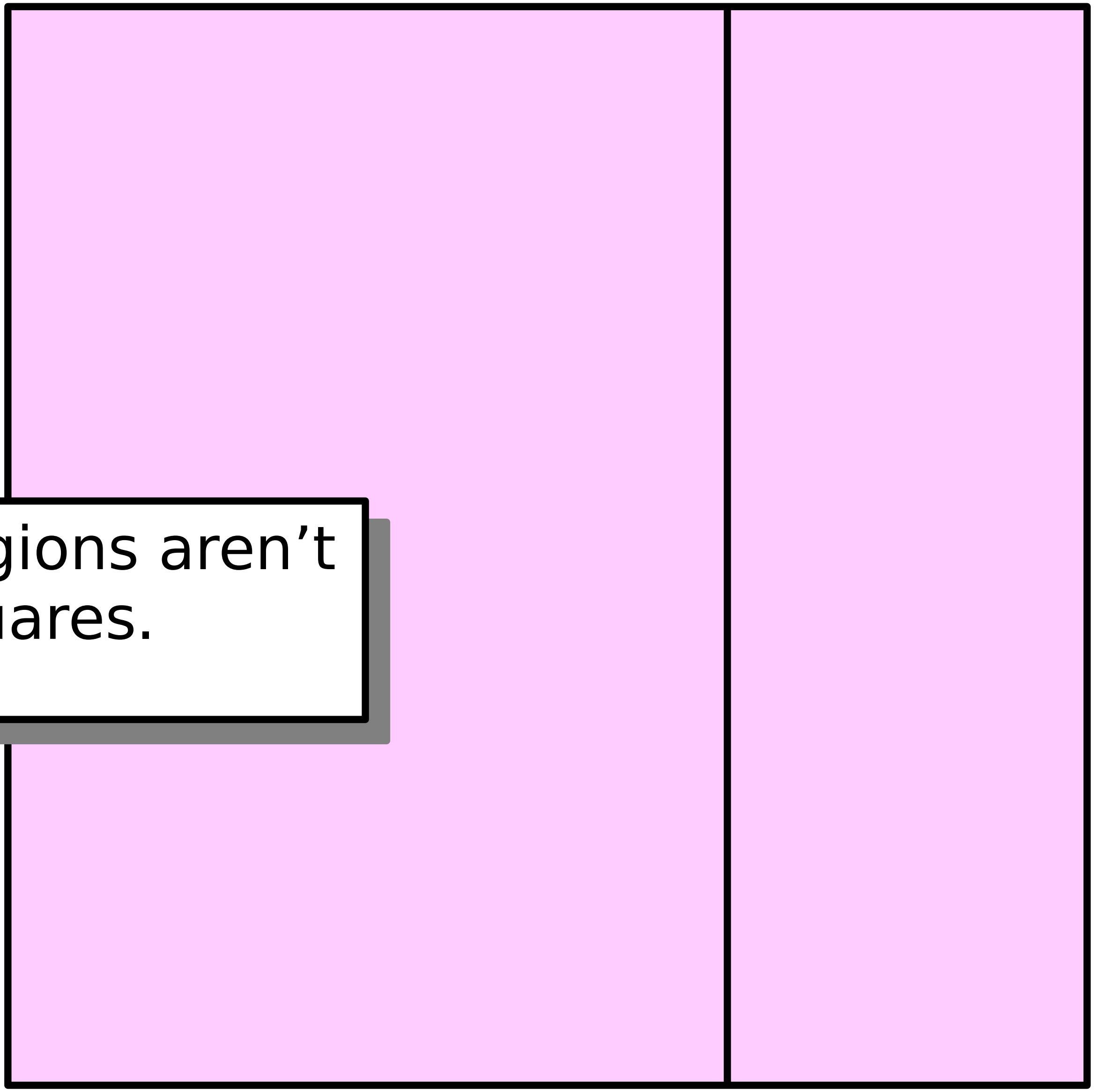
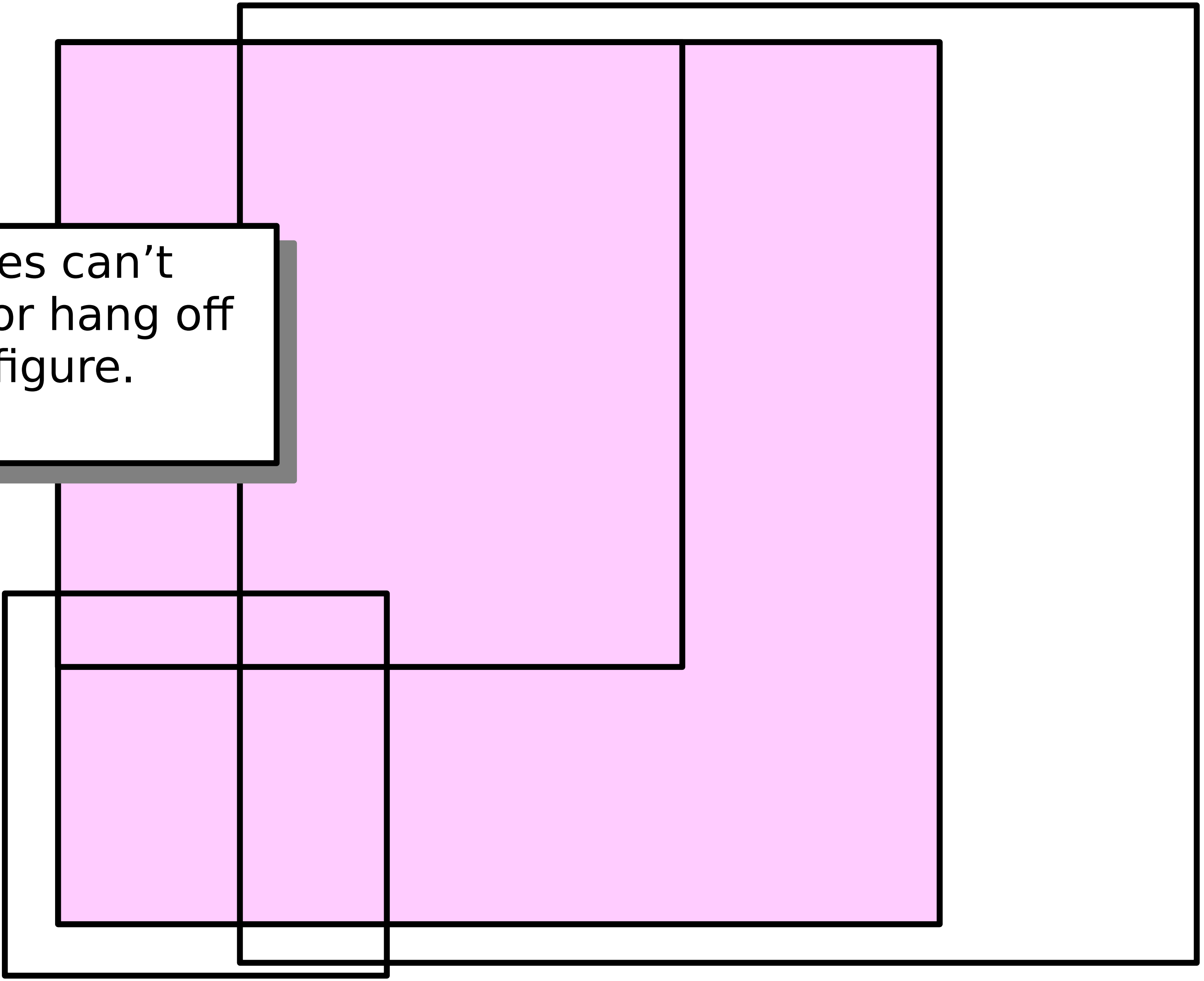


These regions aren’t squares.

Subdividing a Square



Squares can’t overlap or hang off the figure.

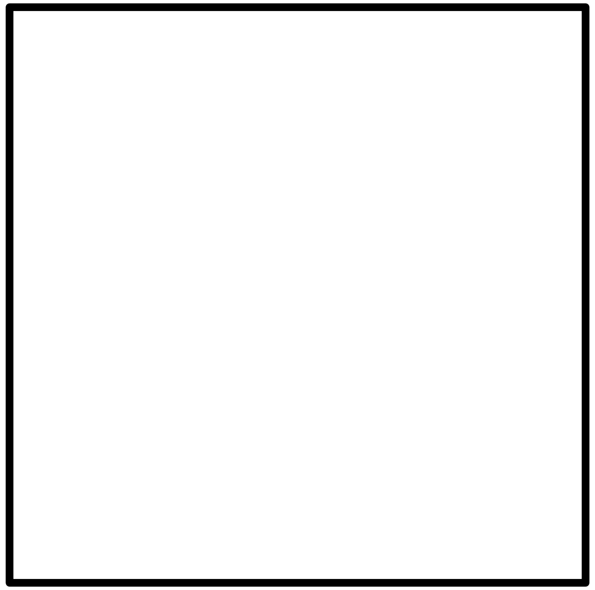
For what values of n can a square be subdivided into n squares?

1 2 3 4 5 6 7 8 9 10 11 12

1 2 3 4 5 6 7 8 9 10 11 12

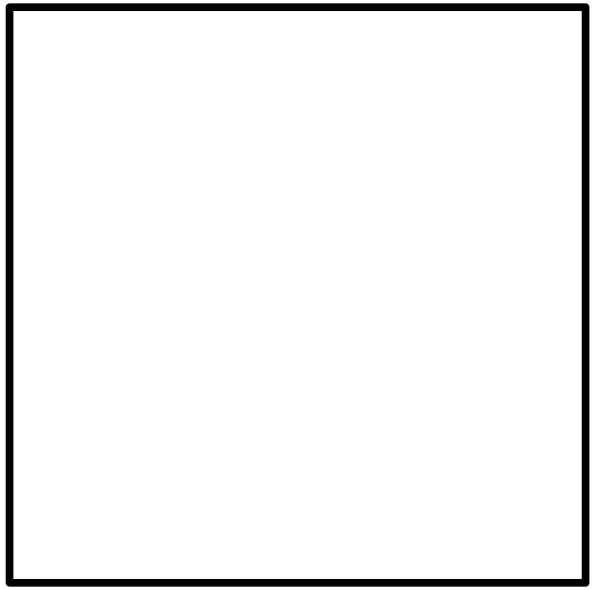
1

1 2 3 4 5 6 7 8 9 10 11 12



1 2 3 4 5 6 7 8 9 10 11 12

Each of the original corners needs to be covered by a corner of the new smaller squares.



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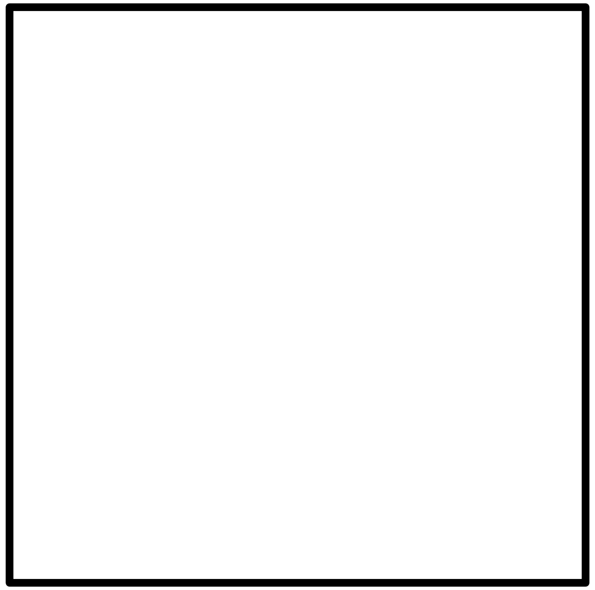

corners: 4

squares: <4

1 2 3 4 5 6 7 8 9 10 11 12

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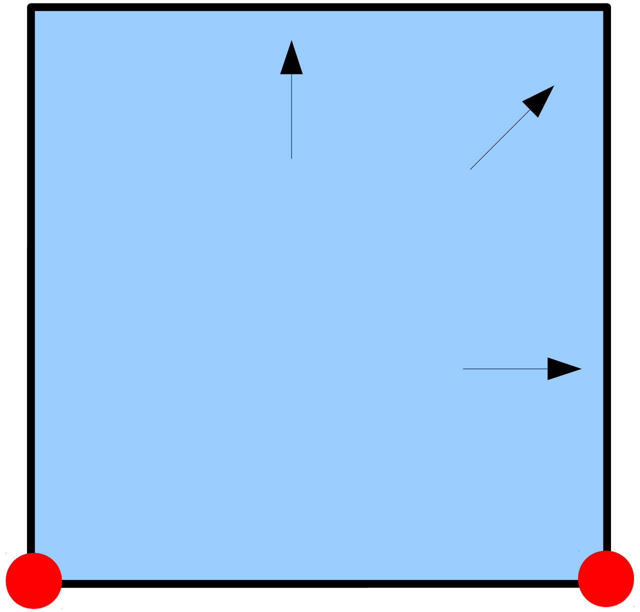
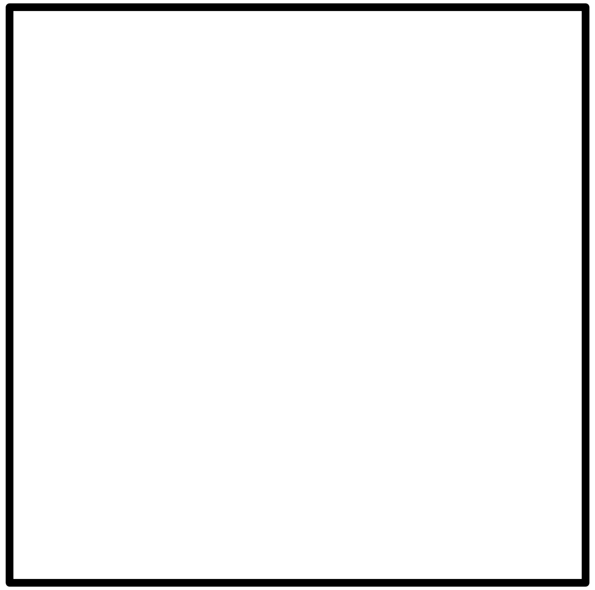


By the pigeonhole principle, at least one smaller square needs to cover at least *two* of the original square’s corners.

1 ~~2~~ ~~3~~ 4 5 6 7 8 9 10 11 12

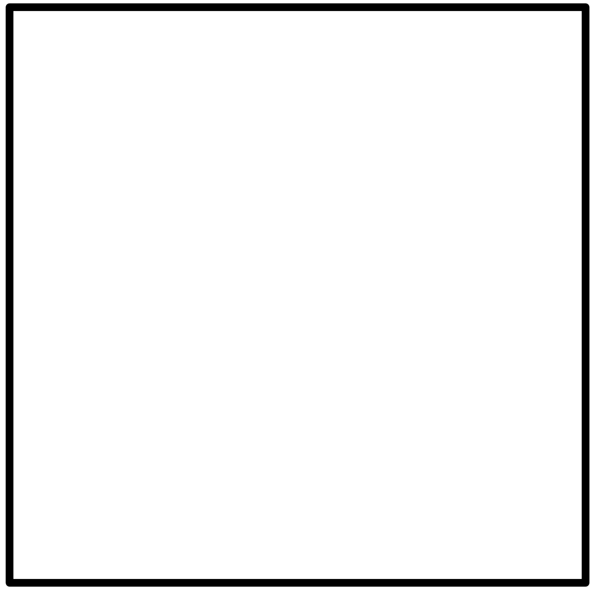


1 ~~2~~ ~~3~~ 4 5 6 7 8 9 10 11 12

1	2
4	3

1 ~~2~~ ~~3~~ 4 5 6 7 8 9 10 11 12

1 ~~2~~ ~~3~~ 4 5 6 7 8 9 10 11 12

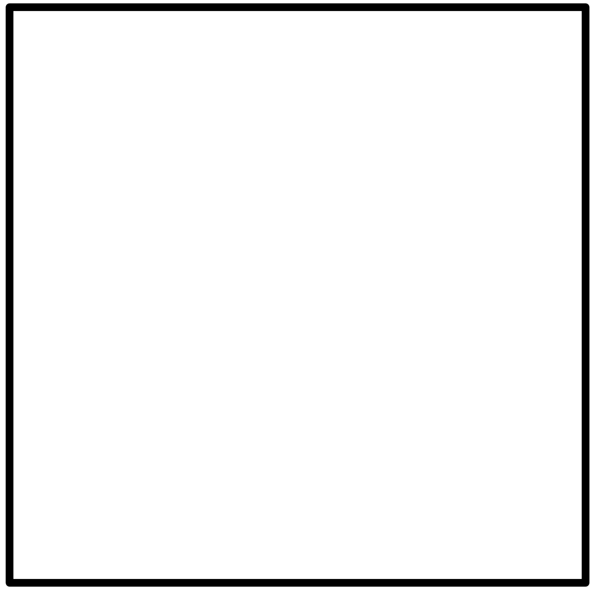


corners: 4

squares: 5

1 ~~2~~ ~~3~~ 4 5 6 7 8 9 10 11 12

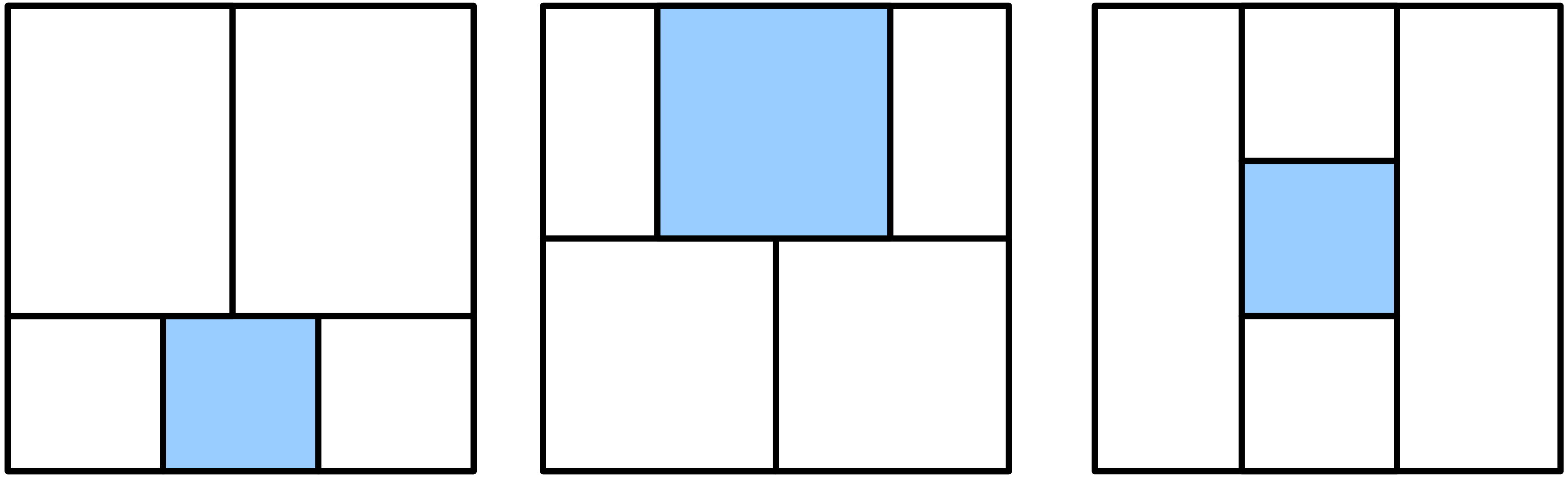
At least one square cannot be covering *any* of the original corners



corners: 4

squares: 5

1 ~~2~~ ~~3~~ 4 5 6 7 8 9 10 11 12



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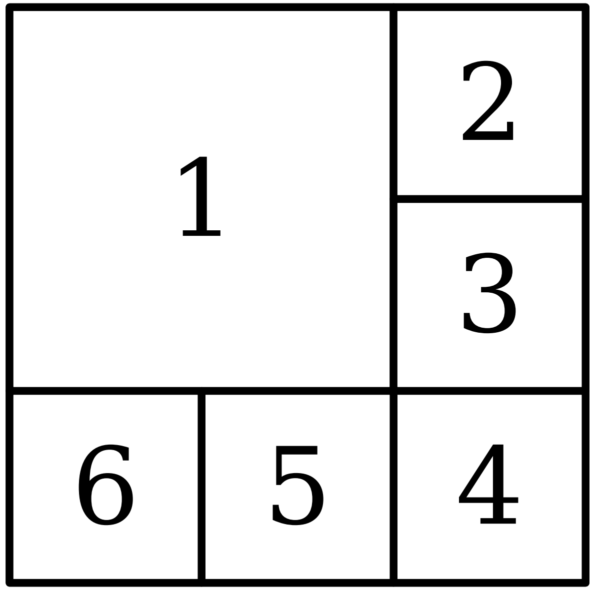

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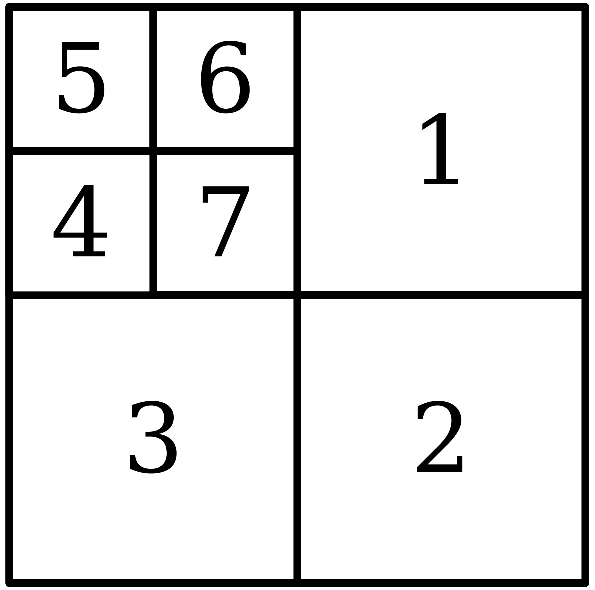

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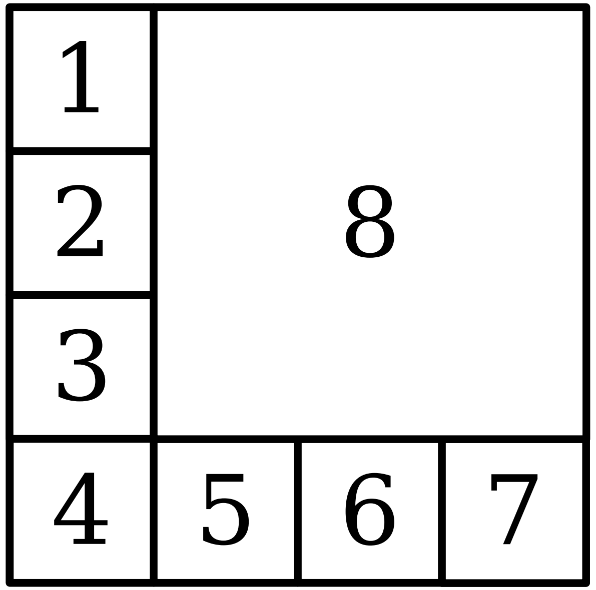
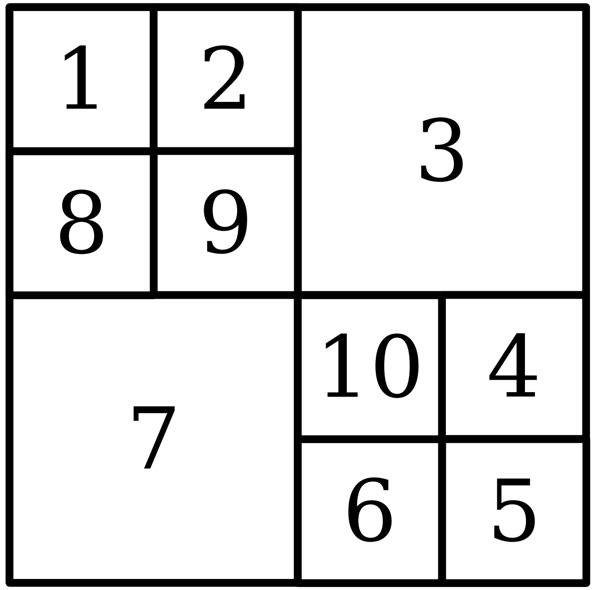


1 ~~2~~ ~~3~~ 4 ~~5~~ 6 7 8 9 10 11 12

1	2	3
8	9	4
7	6	5

1 ~~2~~ ~~3~~ 4 ~~5~~ 6 7 8 9 10 11 12



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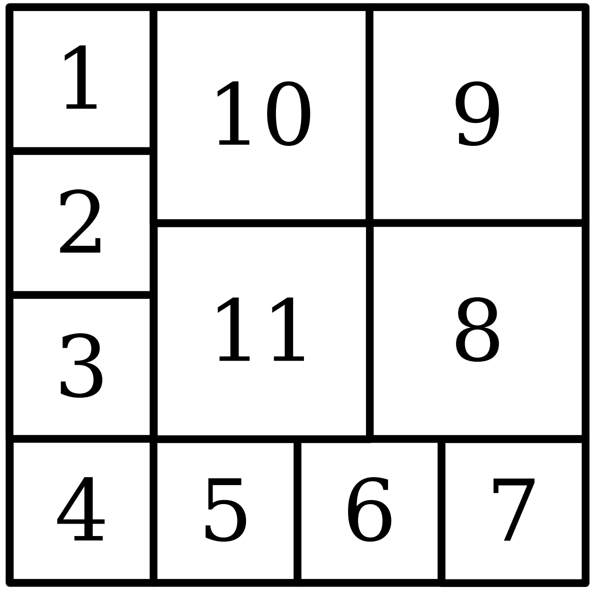

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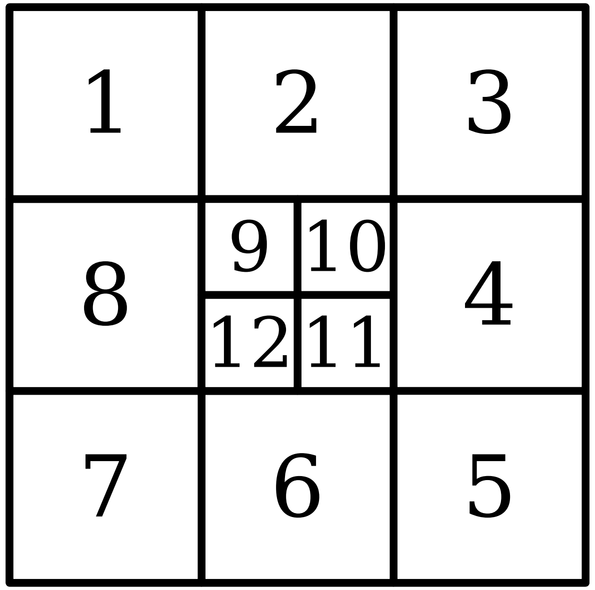

An Insight

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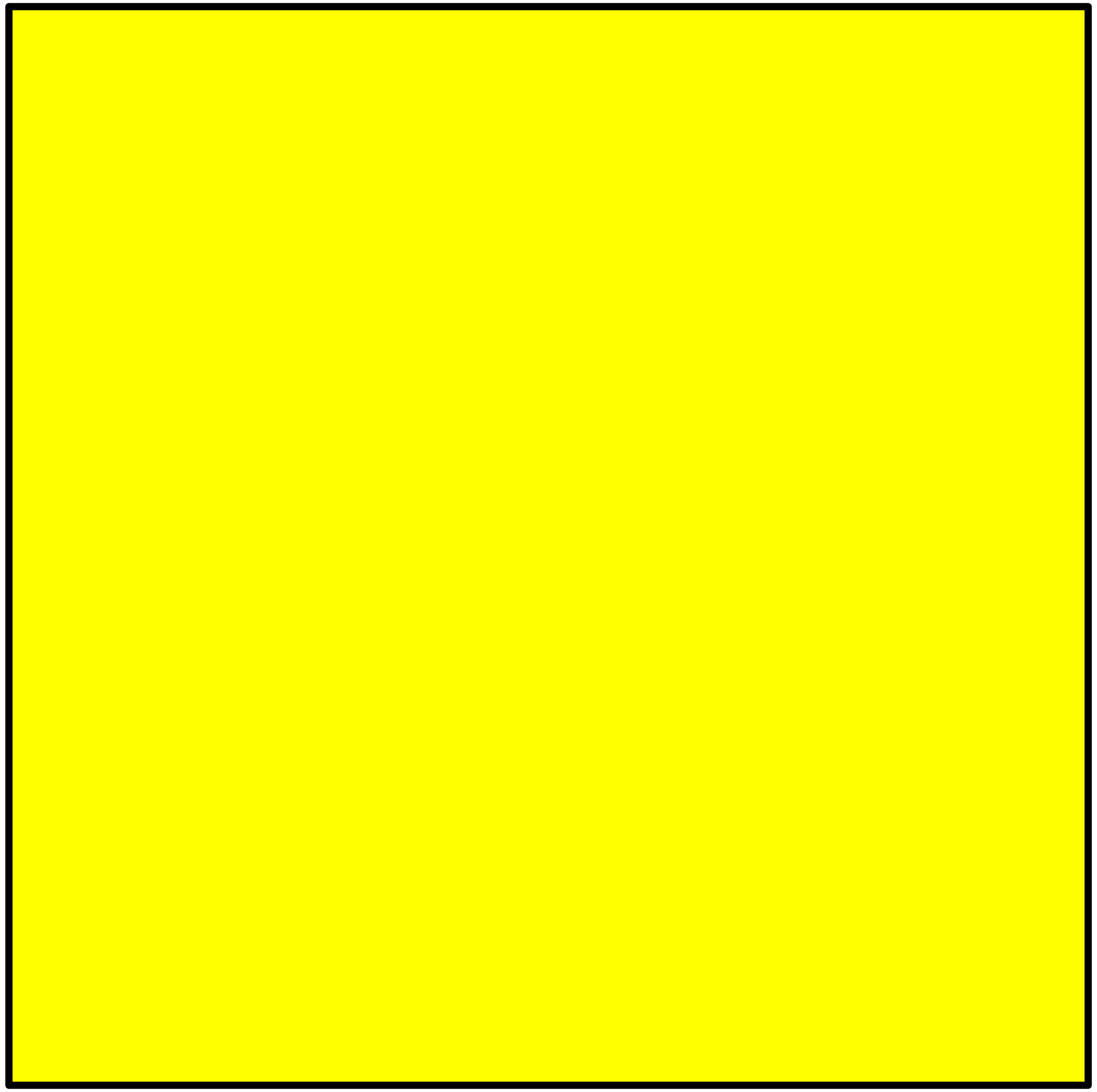

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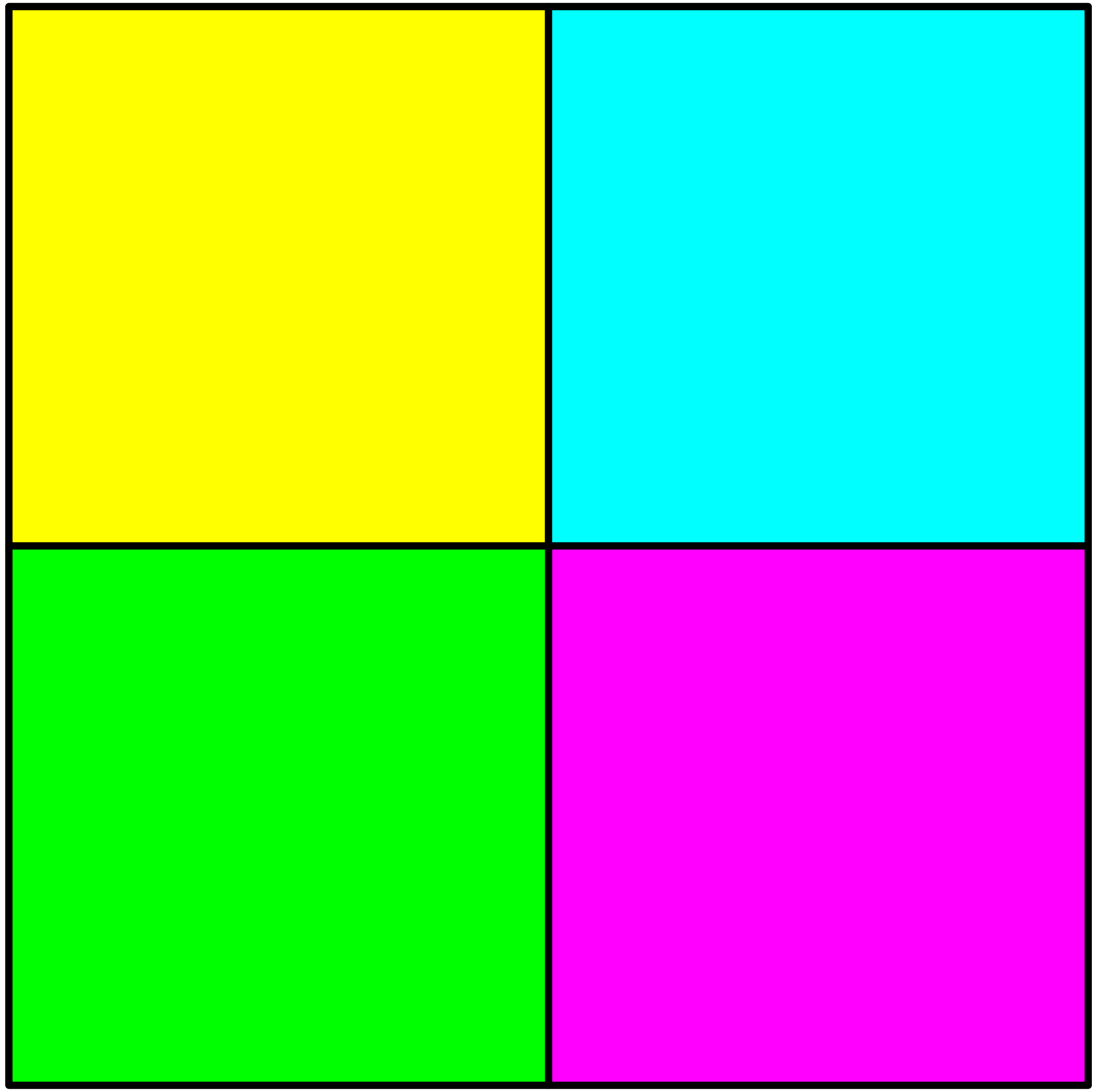

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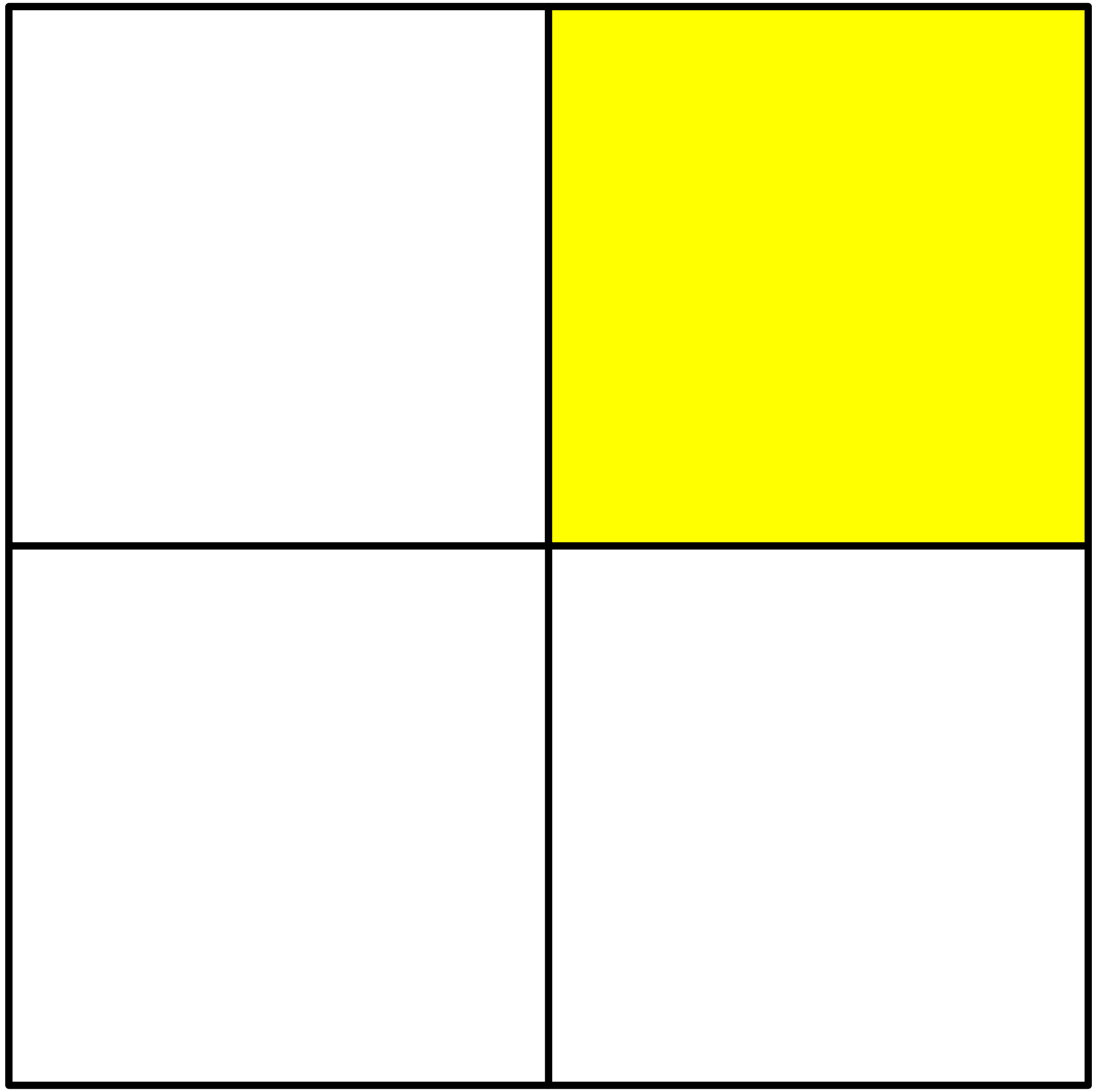

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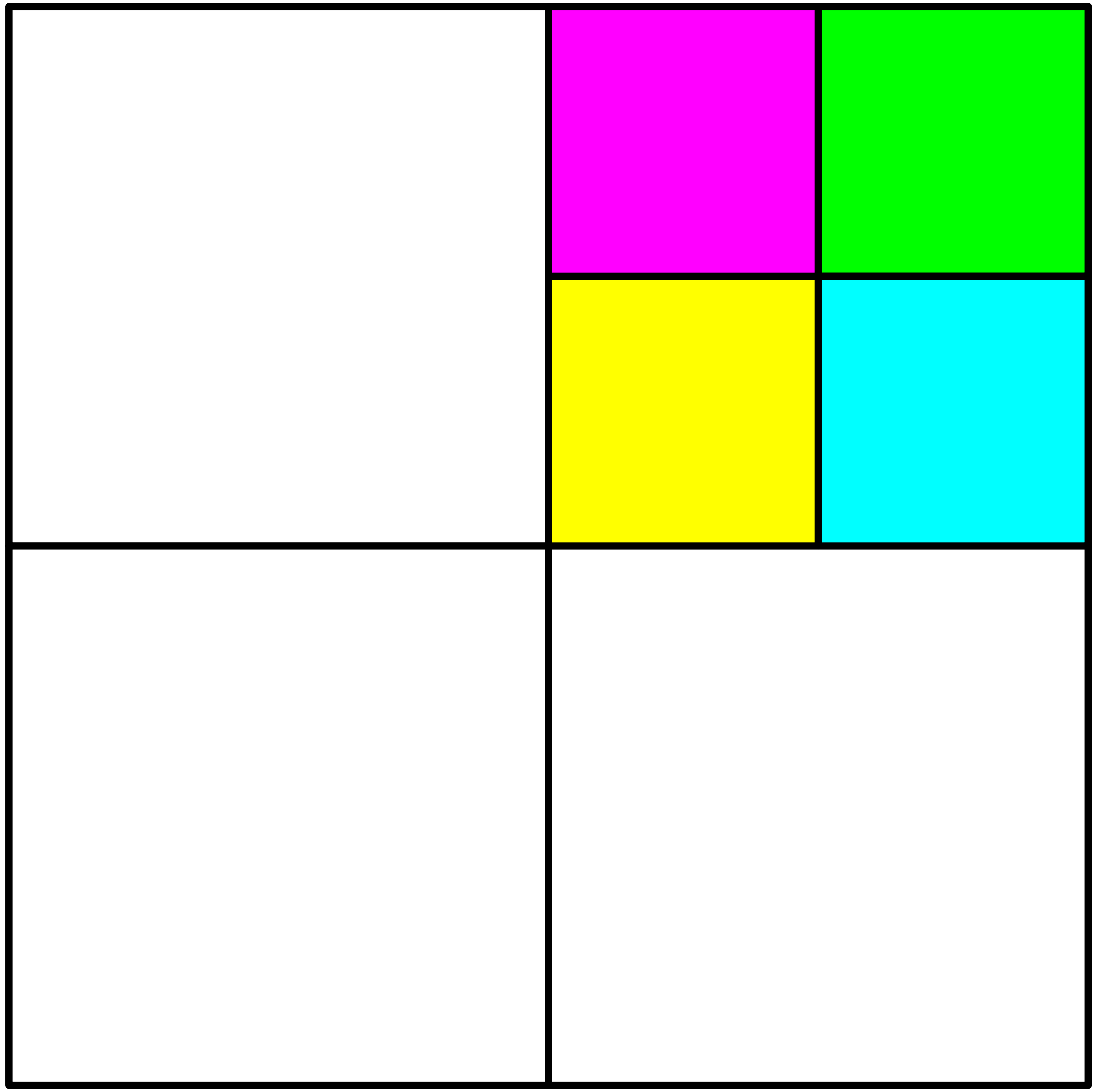

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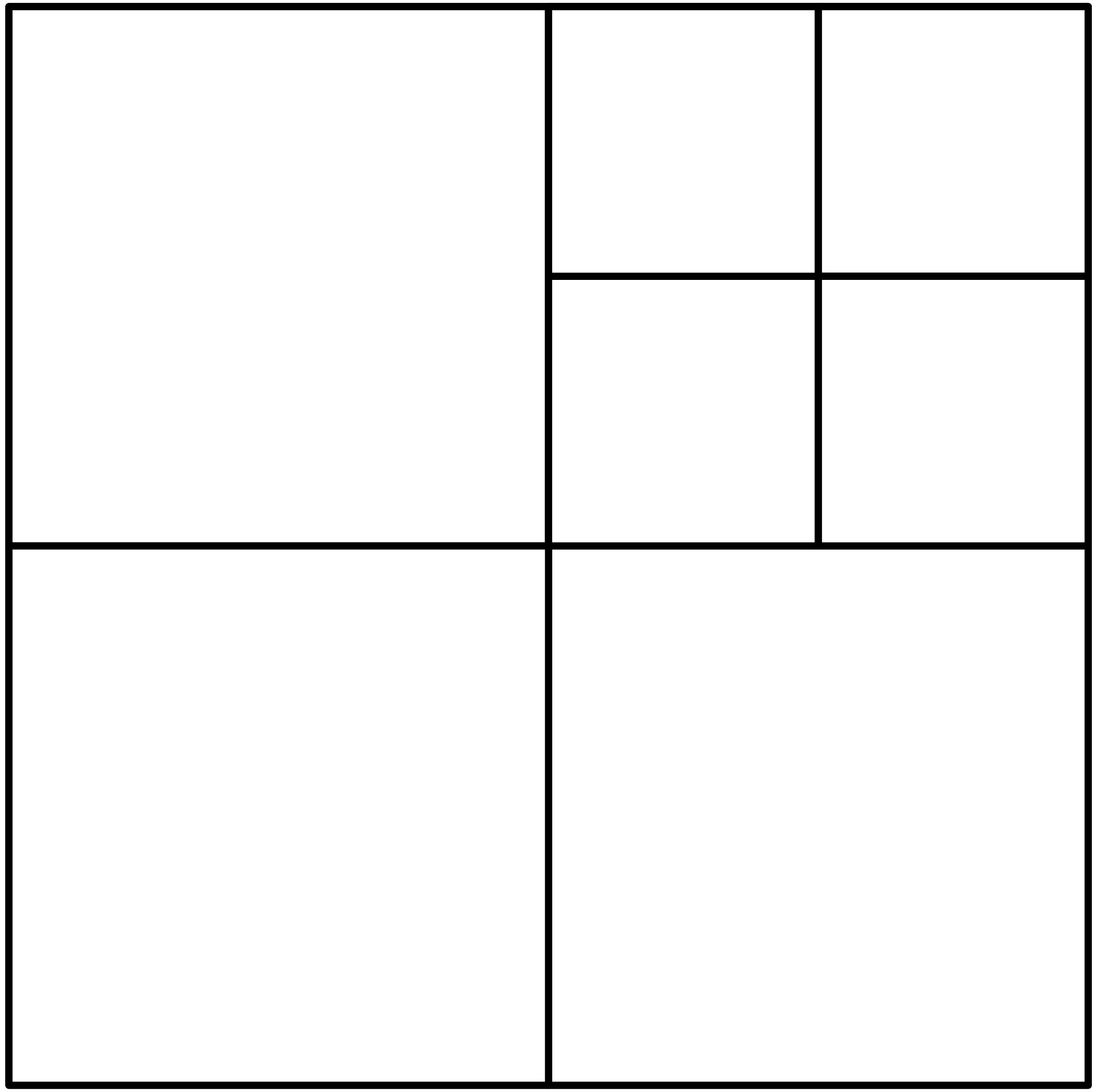

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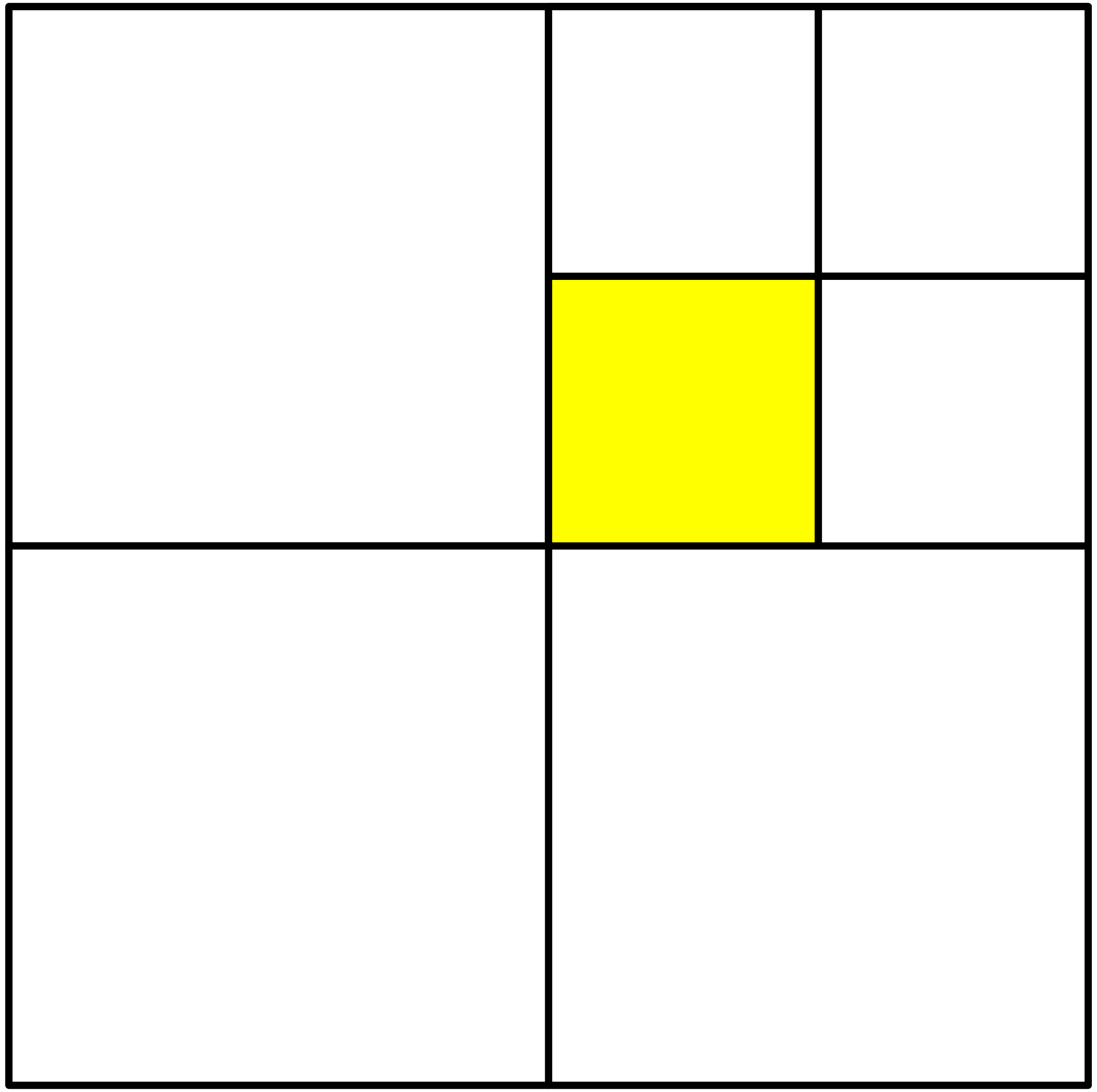

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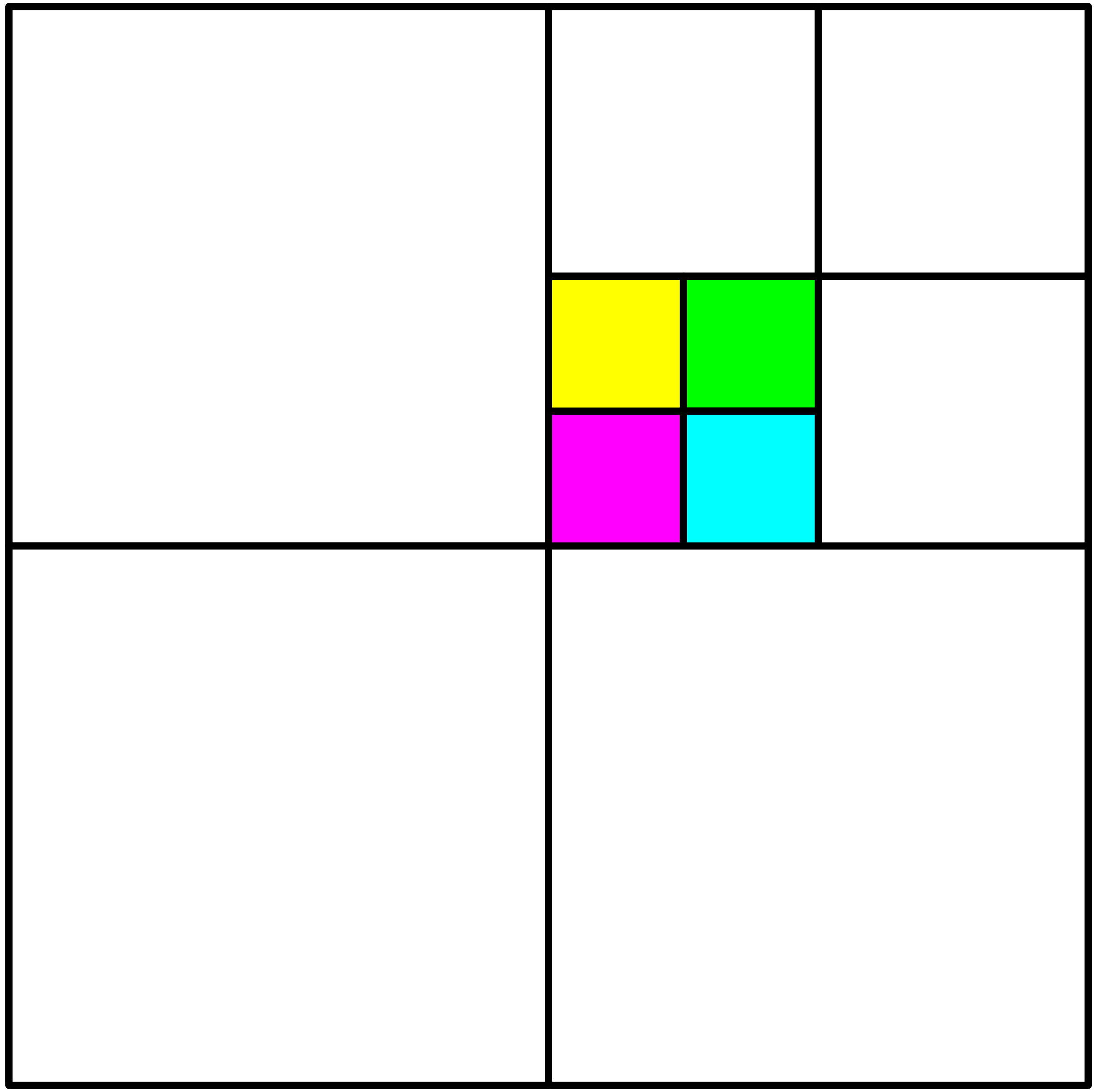

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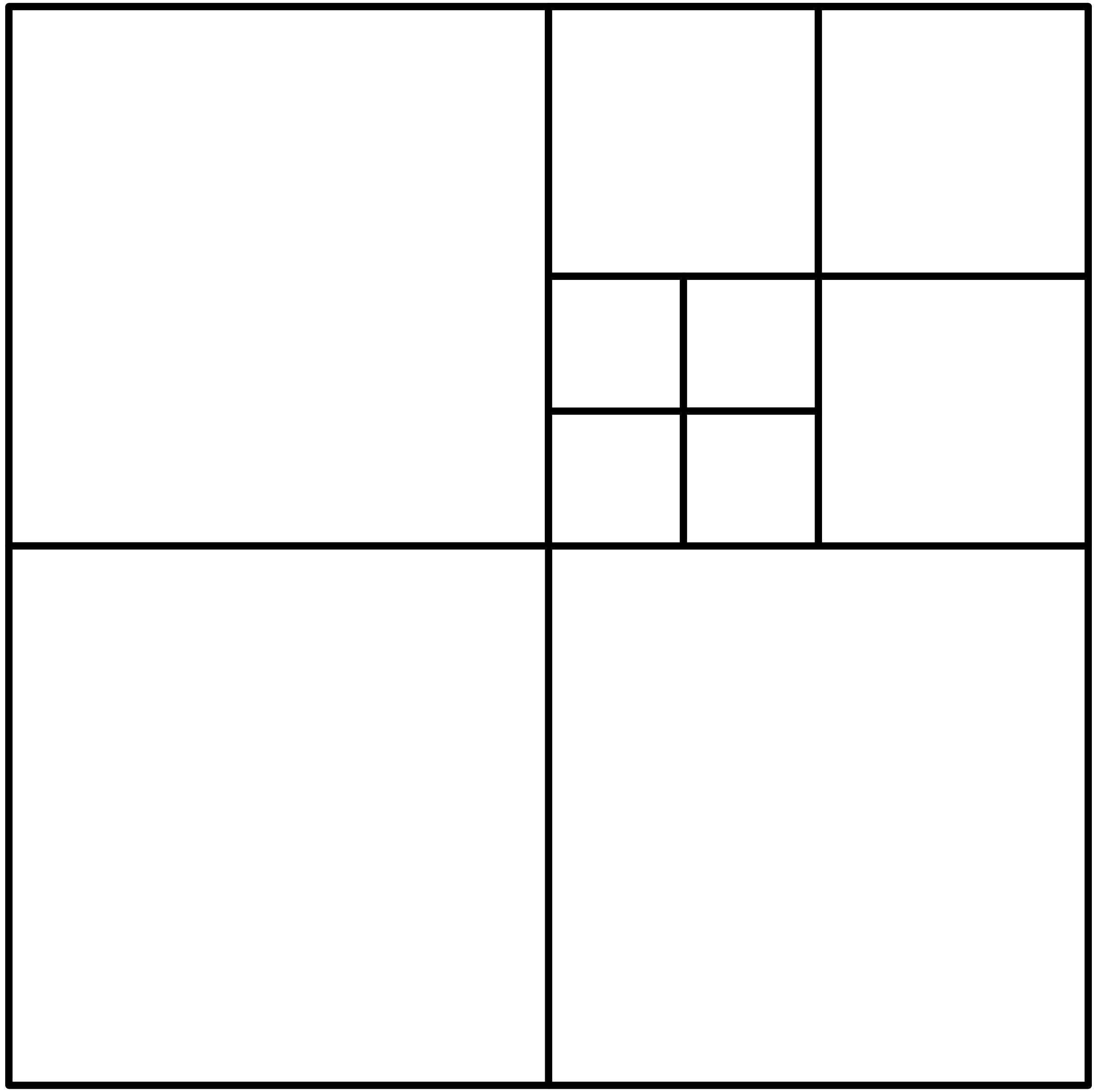

An Insight

- If we can subdivide a square into n squares, we can also subdivide it into $n + 3$ squares.
- Since we can subdivide a bigger square into 6, 7, and 8 squares, we can subdivide a square into n squares for any $n \geq 6$:
  - For multiples of three, start with 6 and keep adding three squares until n is reached.
  - For numbers congruent to one modulo three, start with 7 and keep adding three squares until n is reached.
  - For numbers congruent to two modulo three, start with 8 and keep adding three squares until n is reached.

Theorem: For any $n \geq 6$, there is a way to subdivide a square into n smaller squares.

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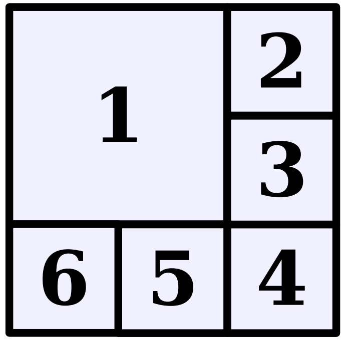

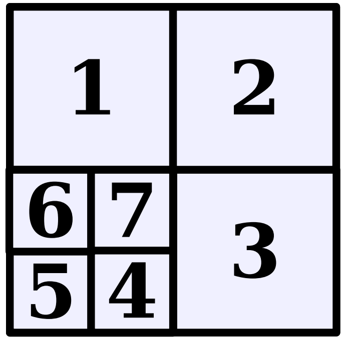

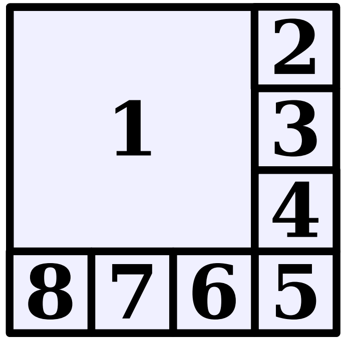

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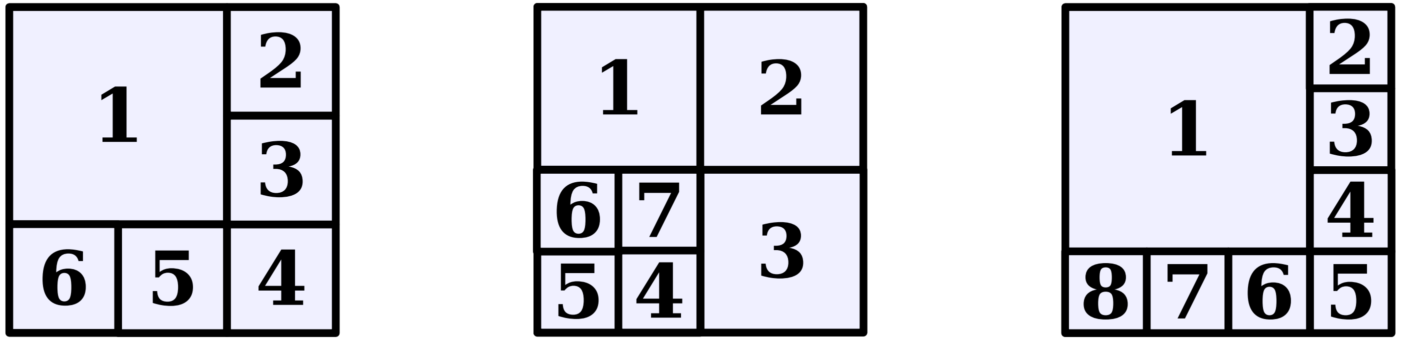

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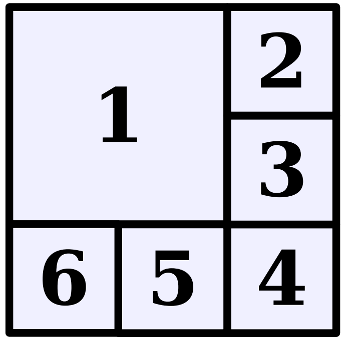

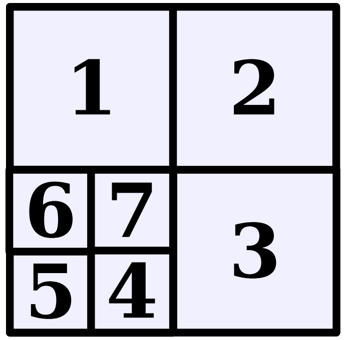

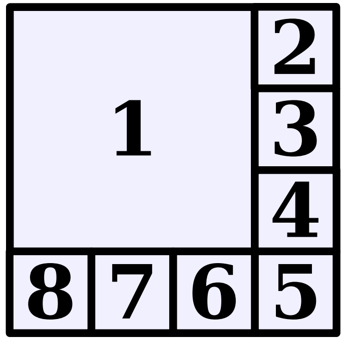

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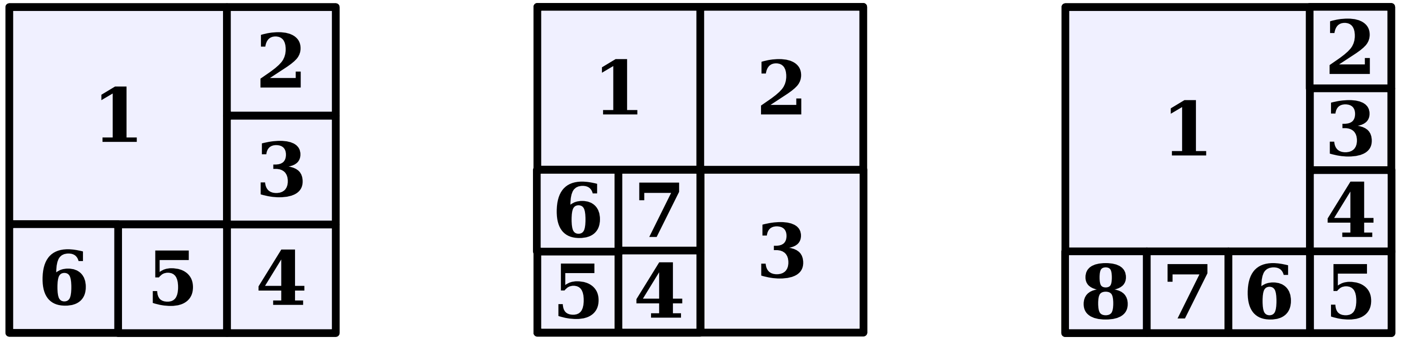

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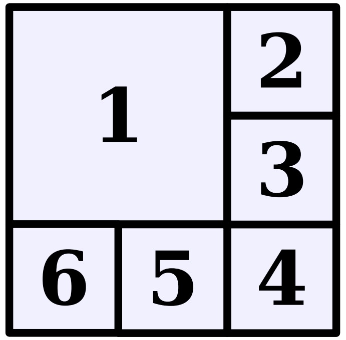

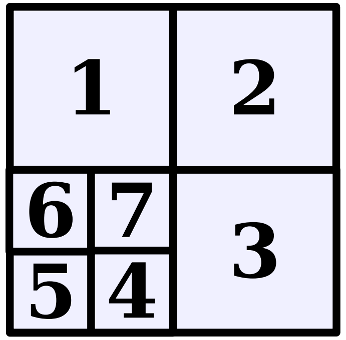

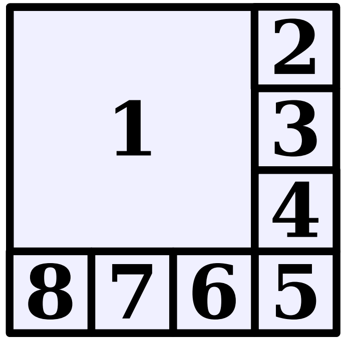

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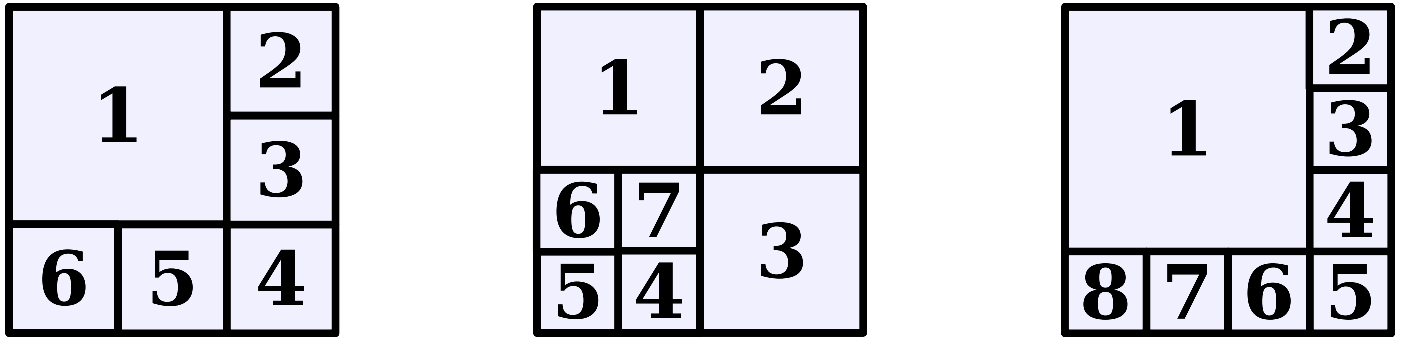

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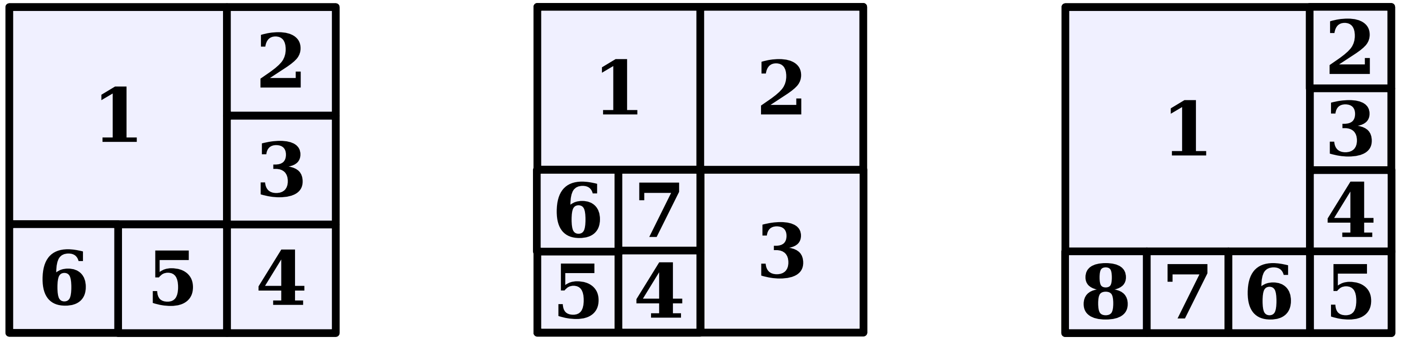

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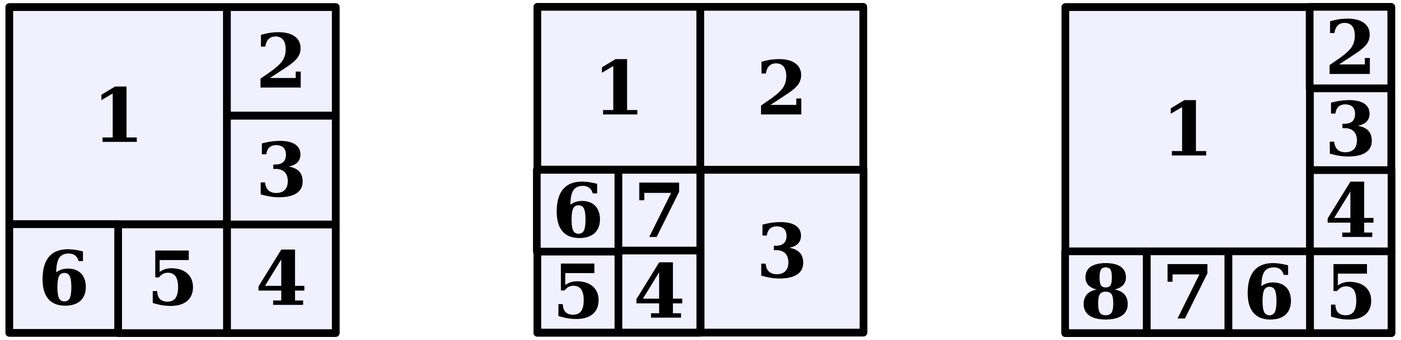

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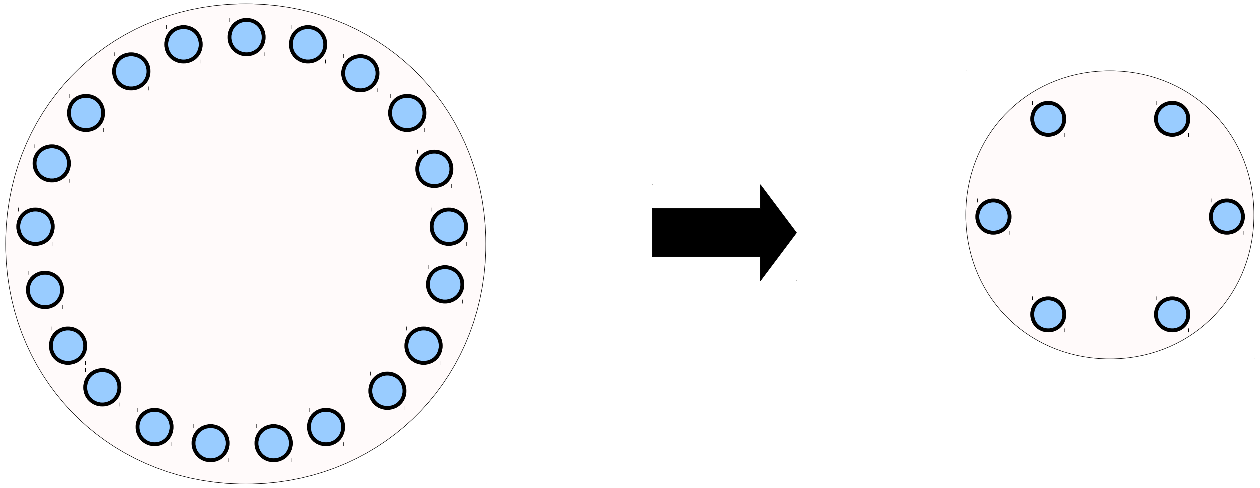
Generalizing Induction

- When doing a proof by induction,
  - feel free to use multiple base cases, and
  - feel free to take steps of sizes other than one.
- If you do, make sure that…
  - … you actually need all your base cases. Avoid redundant base cases that are already covered by a mix of other base cases and your inductive step.
  - … you cover all the numbers you need to cover. Trace out your reasoning and make sure all the numbers you need to cover really are covered.
- As with a proof by cases, you don't need to separately prove you've covered all the options. We trust you. 😃

More on Square Subdivisions

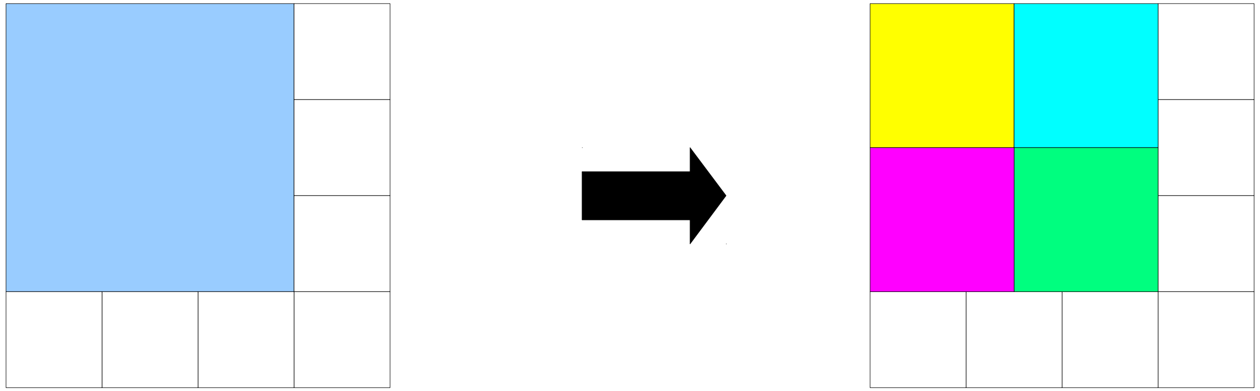
- There are a ton of interesting questions that come up when trying to subdivide a rectangle or square into smaller squares.
- In fact, one of the major players in early graph theory (William Tutte) got his start playing around with these problems.
- Good starting resource: this Numberphile video on ***Squaring the Square***.

An Observation



Start with larger clique

Get to smaller clique



Start with fewer squares

Get to more squares

Following the Rules

- When working with square subdivisions, our predicate looked like this:

  $P(n)$ is "***there exists*** a way to subdivide a square into n squares."

- When working with cliques, our predicate looked like this:

  $P(n)$ is "***for any*** coloring of a $3n!$-clique, there is a monochrome triangle."

- With squares, the quantifier is ∃. With cliques, the first quantifier is ∀.
- This fundamentally changes the "feel" of induction.

Build Up with ∃

- In the case of squares, in our inductive step, we prove

  If

  there exists a subdivision into k squares,

  then

  there exists a subdivision into $k+3$ squares.

- Assuming the antecedent gives us a concrete subdivision into k squares.
- Proving the consequent means finding some way to subdivide in to $k+3$ squares.
- The inductive step goal is to "***build up***:" start with a smaller number of squares, and somehow work out what to do to get a larger number of squares.

Build Down with ∀

- In the case of cliques, in our inductive step, we prove

  If

  for all colorings of a $3k!$-clique, there's a mono. tri.

  then

  for all colorings of a $3(k+1)!$-clique, there's a mono. tri.
- Assuming the antecedent means once we find a k-colored $3k!$-clique, we get a monochrome triangle.
- Proving the consequent means picking an arbitrary coloring of a $3(k+1)!$-clique, then trying to find a triangle in it.
- The inductive step goal is to "***build down***:" start with a larger clique, then find a way to turn it into a smaller clique.

Complete Induction

Let P be some predicate. The ***principle of complete induction*** states that if

If it starts true…

$P(0)$ is true

and

…and it stays true…

for all $k \in \mathbb{N}$, if $P(0)$, …, and $P(k)$ are true, then $P(k+1)$ is true

then

$\forall n \in \mathbb{N}.\ P(n)$

…then it's always true.

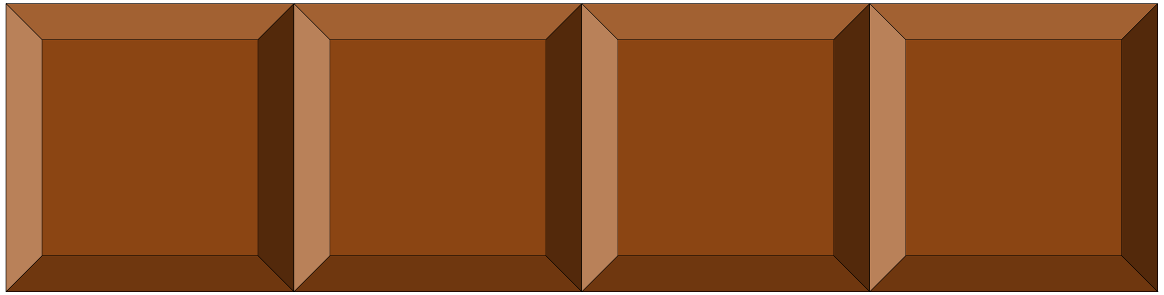
Mathematical Induction

- You can write proofs using the principle of mathematical induction as follows:
  - Define some predicate $P(n)$ to prove by induction on n.
  - Choose and prove a base case (probably, but not always, $P(0)$).
  - Pick an arbitrary $k \in \mathbb{N}$ and assume that $P(k)$ is true.
  - Prove $P(k+1)$.
  - Conclude that $P(n)$ holds for all $n \in \mathbb{N}$.

Complete Induction

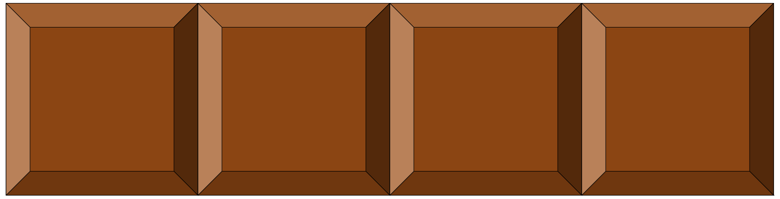
- You can write proofs using the principle of ***complete*** induction as follows:
  - Define some predicate $P(n)$ to prove by induction on n.
  - Choose and prove a base case (probably, but not always, $P(0)$).
  - Pick an arbitrary $k \in \mathbb{N}$ and assume that **$P(0)$, $P(1)$, $P(2)$, …, and $P(k)$** are all true.
  - Prove $P(k+1)$.
  - Conclude that $P(n)$ holds for all $n \in \mathbb{N}$.

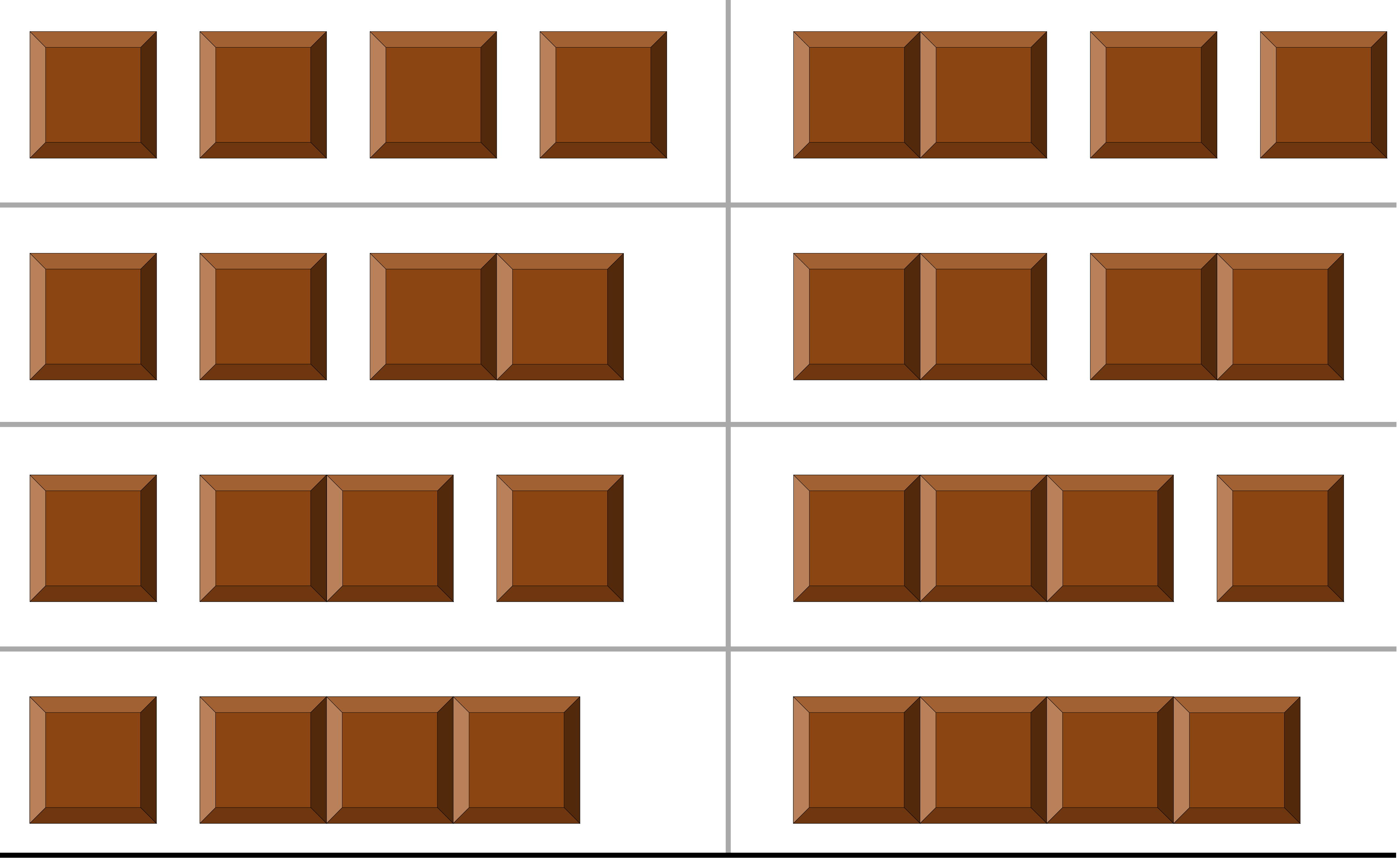
An Example: ***Eating a Chocolate Bar***



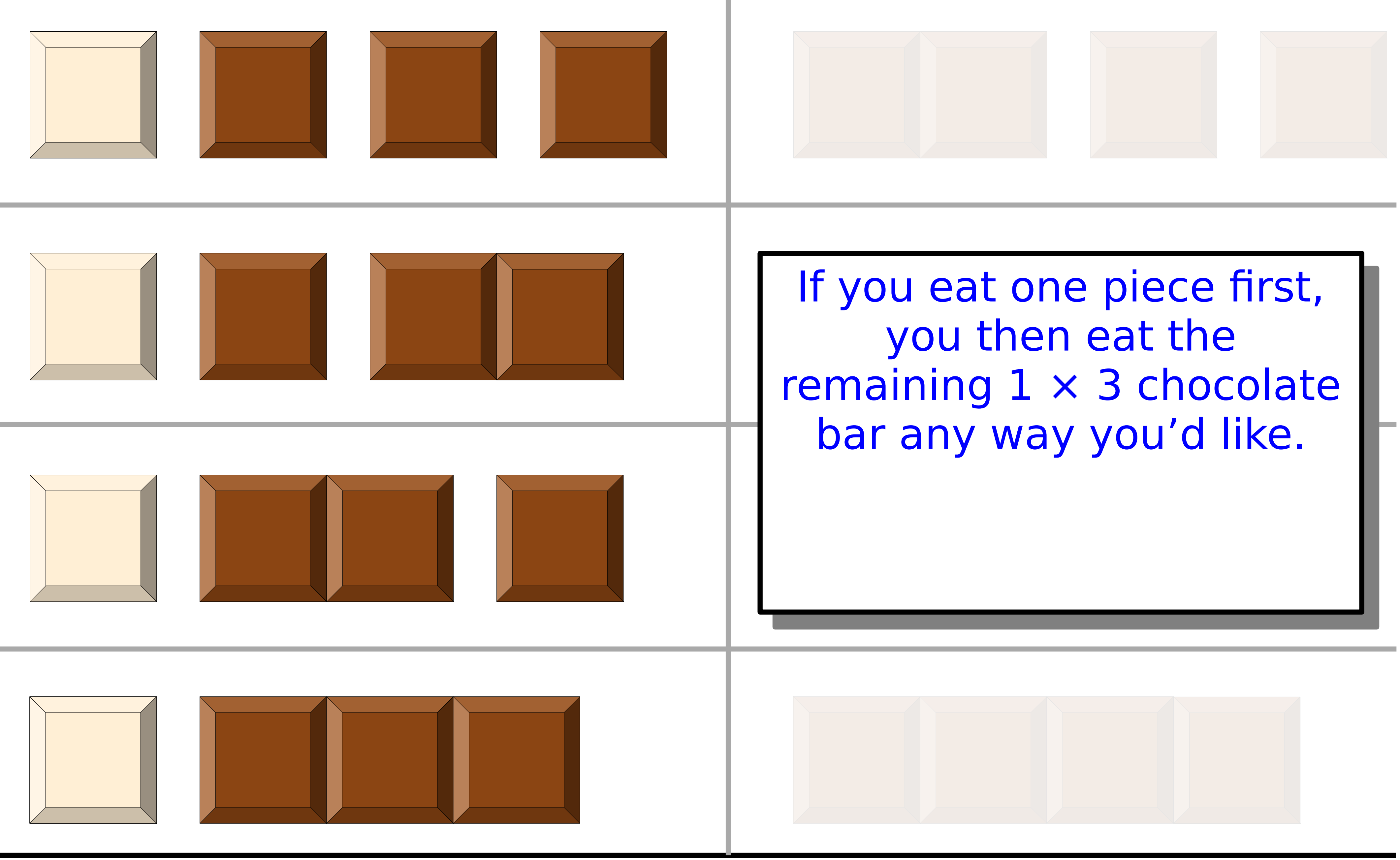
Eating a Chocolate Bar

- You have a $1 \times n$ chocolate bar subdivided into 1×1 squares.
- You eat the chocolate bar from left to right by breaking off one or more squares and eating them in one (possibly enormous) bite.
- How many ways can you eat a…
  - 1×1 chocolate bar?
  - 1×2 chocolate bar?
  - 1×3 chocolate bar?
  - 1×4 chocolate bar?



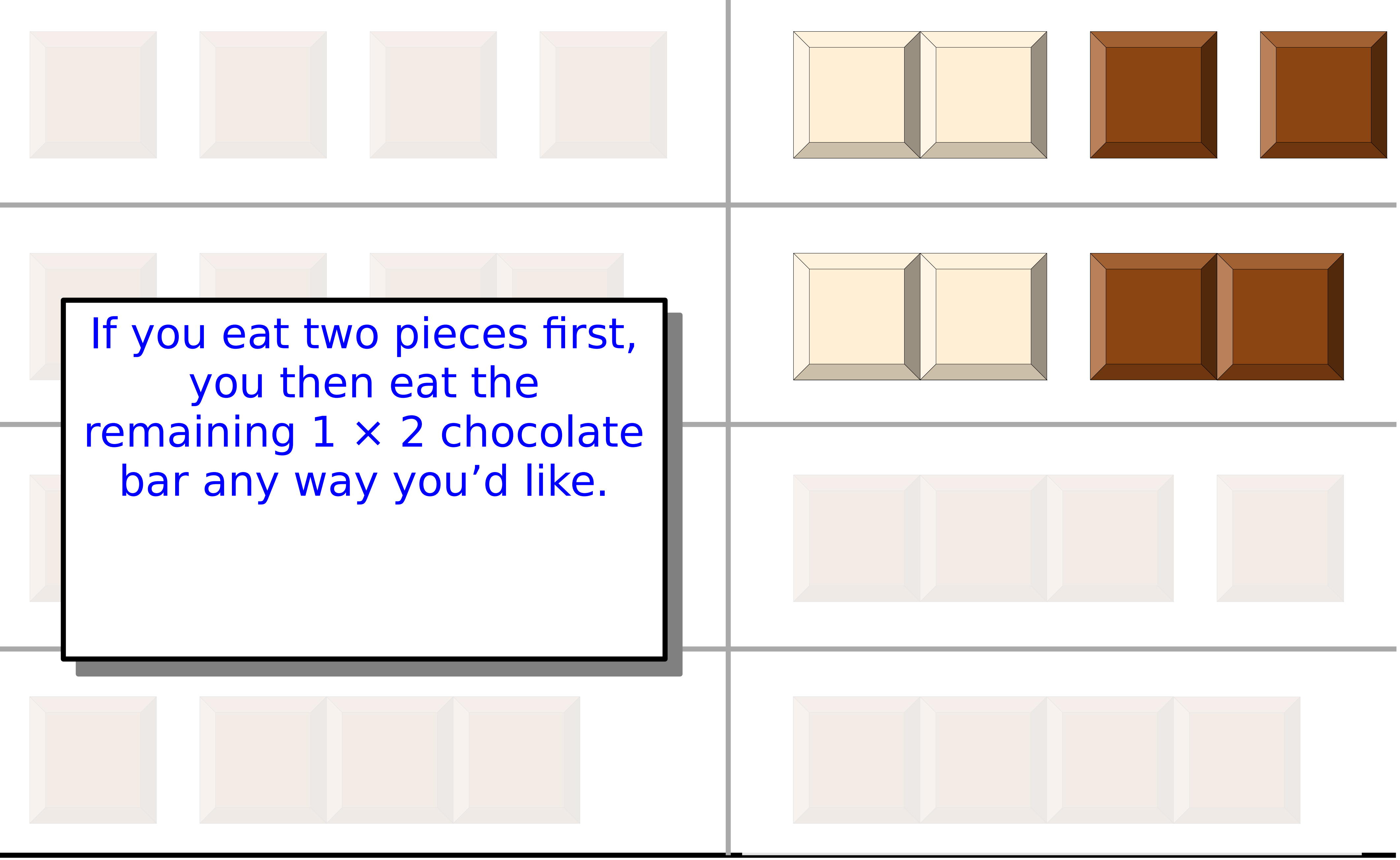


There are eight ways to eat a 1 × 4 chocolate bar.



If you eat one piece first, you then eat the remaining 1 × 3 chocolate bar any way you’d like.

There are eight ways to eat a 1 × 4 chocolate bar.



If you eat two pieces first, you then eat the remaining 1 × 2 chocolate bar any way you’d like.

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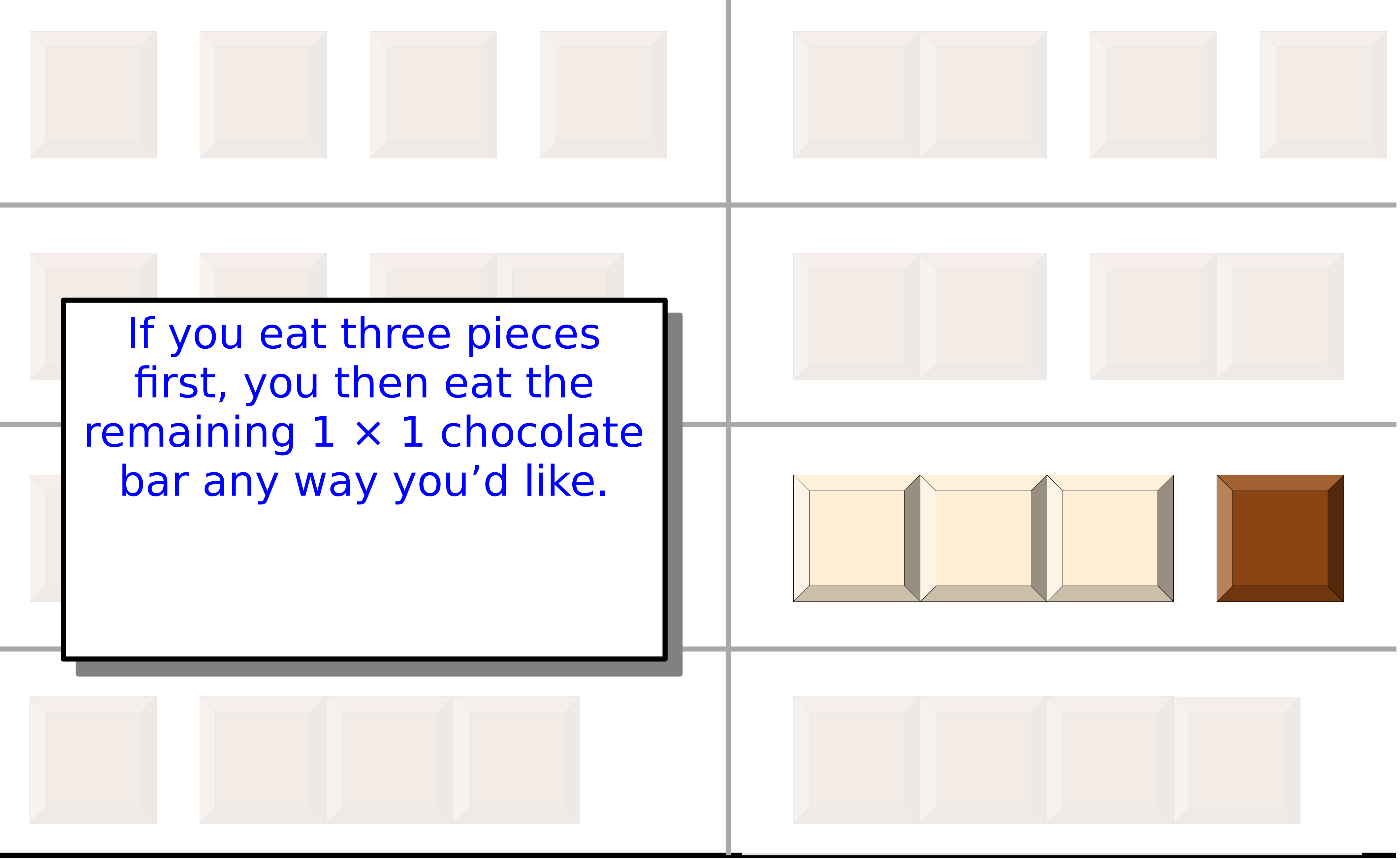
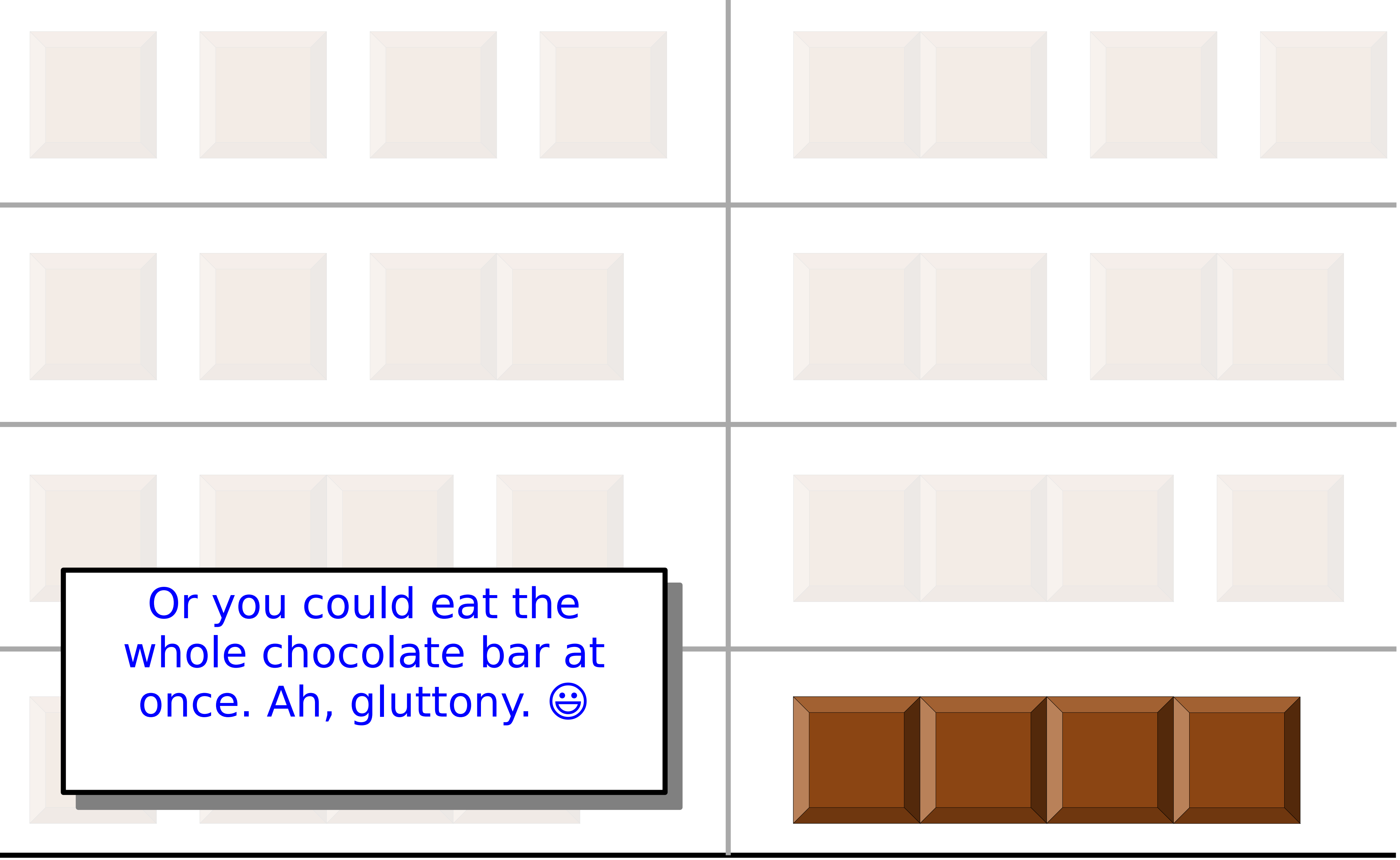


If you eat three pieces first, you then eat the remaining 1 × 1 chocolate bar any way you’d like.

There are eight ways to eat a 1 × 4 chocolate bar.



Or you could eat the whole chocolate bar at once. Ah, gluttony. 😃

There are eight ways to eat a 1 × 4 chocolate bar.

Eating a Chocolate Bar

- There's…
  - 1 way to eat a 1 × 1 chocolate bar,
  - 2 ways to eat a 1 × 2 chocolate bar,
  - 4 ways to eat a 1 × 3 chocolate bar, and
  - 8 ways to eat a 1 × 4 chocolate bar.
- ***Our guess:*** There are 2^{n-1} ways to eat a $1 \times n$ chocolate bar for any natural number $n \geq 1$.
- And we think it has something to do with this insight: we eat the bar either by
  - eating the whole thing in one bite, or
  - eating some piece of size k, then eating the remaining $n - k$ pieces however we'd like.
- Let's formalize this!

Theorem: For any natural number $n \geq 1$, there are exactly 2^{n-1} ways to eat a $1 \times n$ chocolate bar from left to right.

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As our base case, we prove $P(1)$, that there is exactly $2^{1-1} = 1$ way to eat a 1×1 chocolate bar from left to right.

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There are two options for how to eat the bar.

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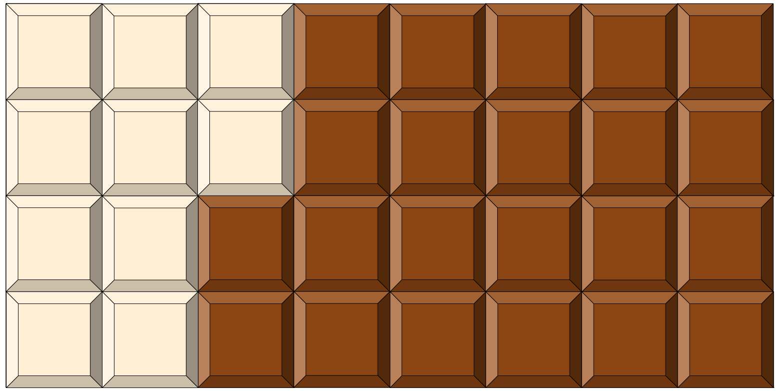
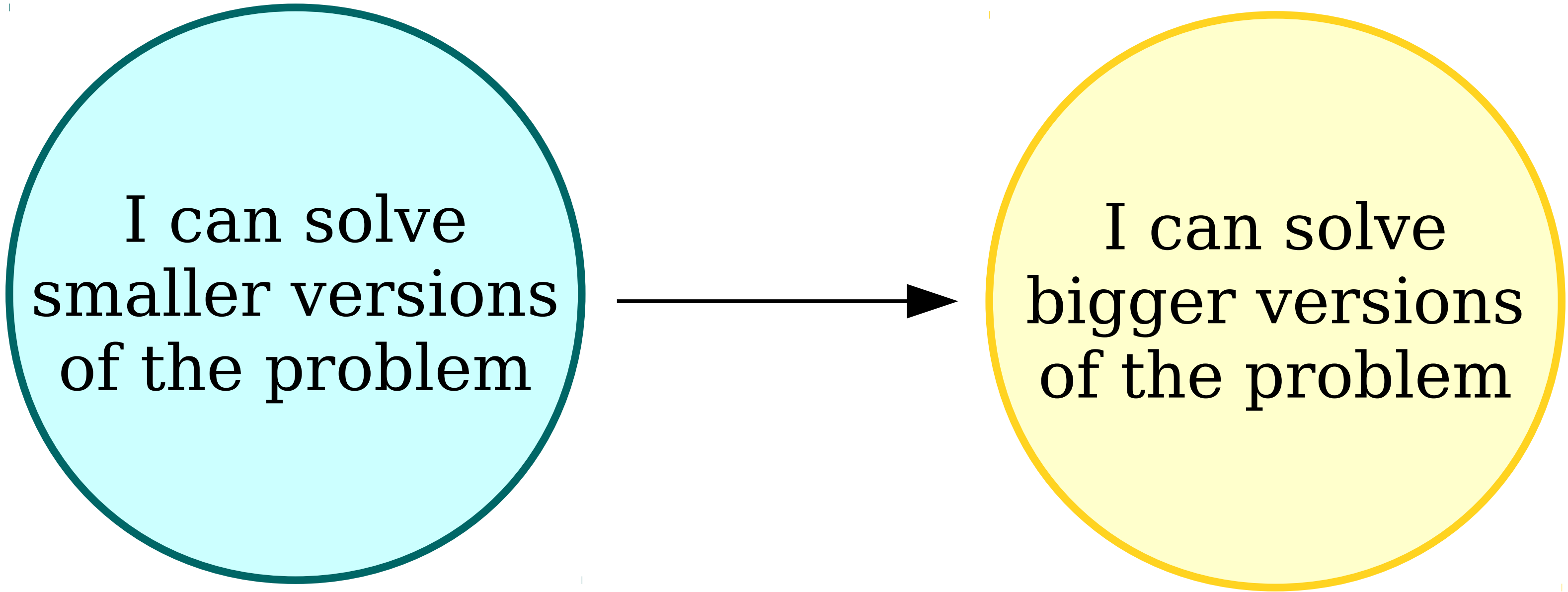
More on Chocolate Bars

- Imagine you have an $m \times n$ chocolate bar. Whenever you eat a square, you have to eat all squares above it and to the left.
- How many ways are there to eat the chocolate bar?



- ***Open Problem:*** Find a non-recursive exact formula for this number, or give an approximation whose error drops to zero as m and n tend toward infinity.

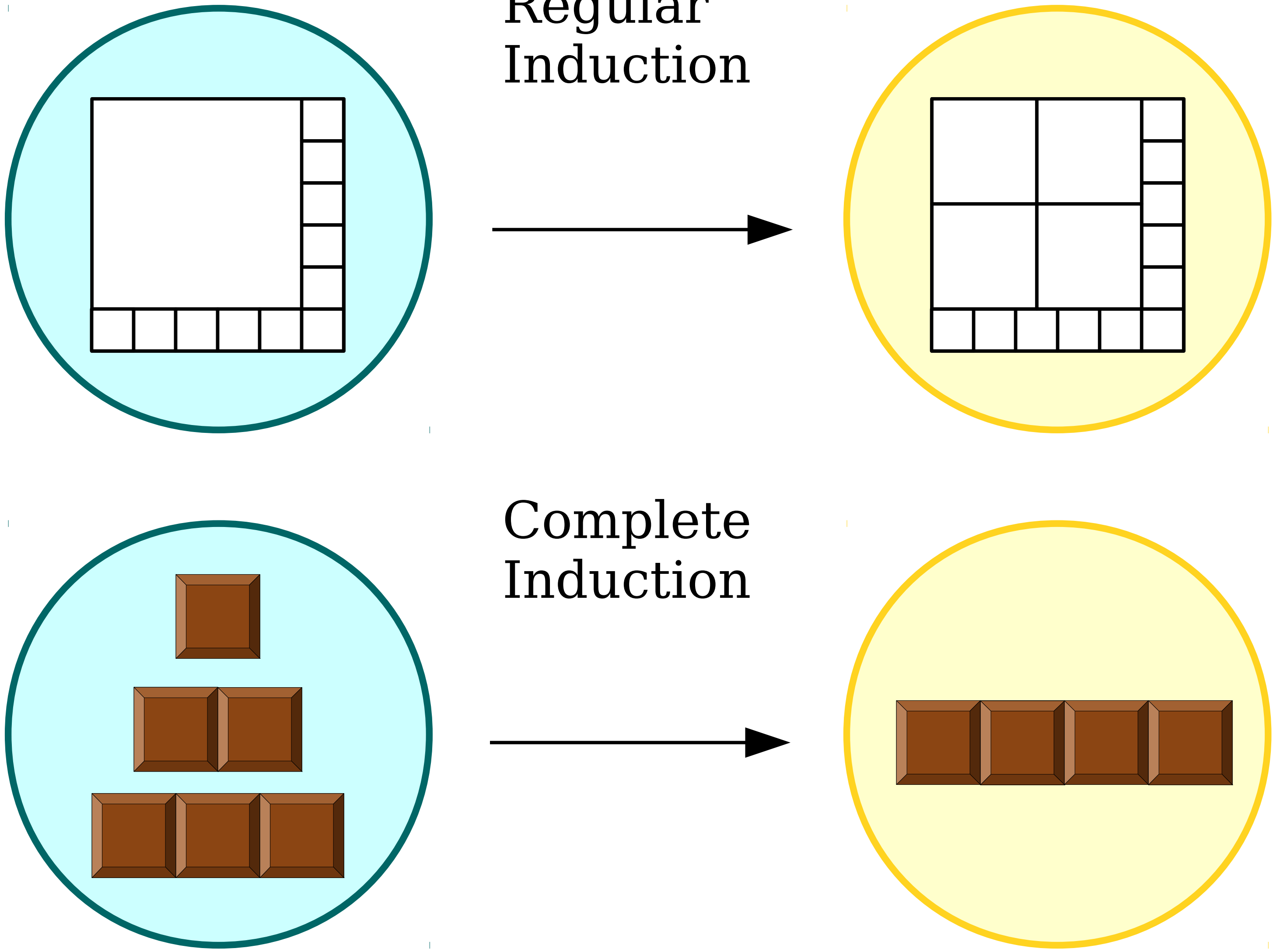
Induction vs. Complete Induction



I can solve smaller versions of the problem

I can solve bigger versions of the problem

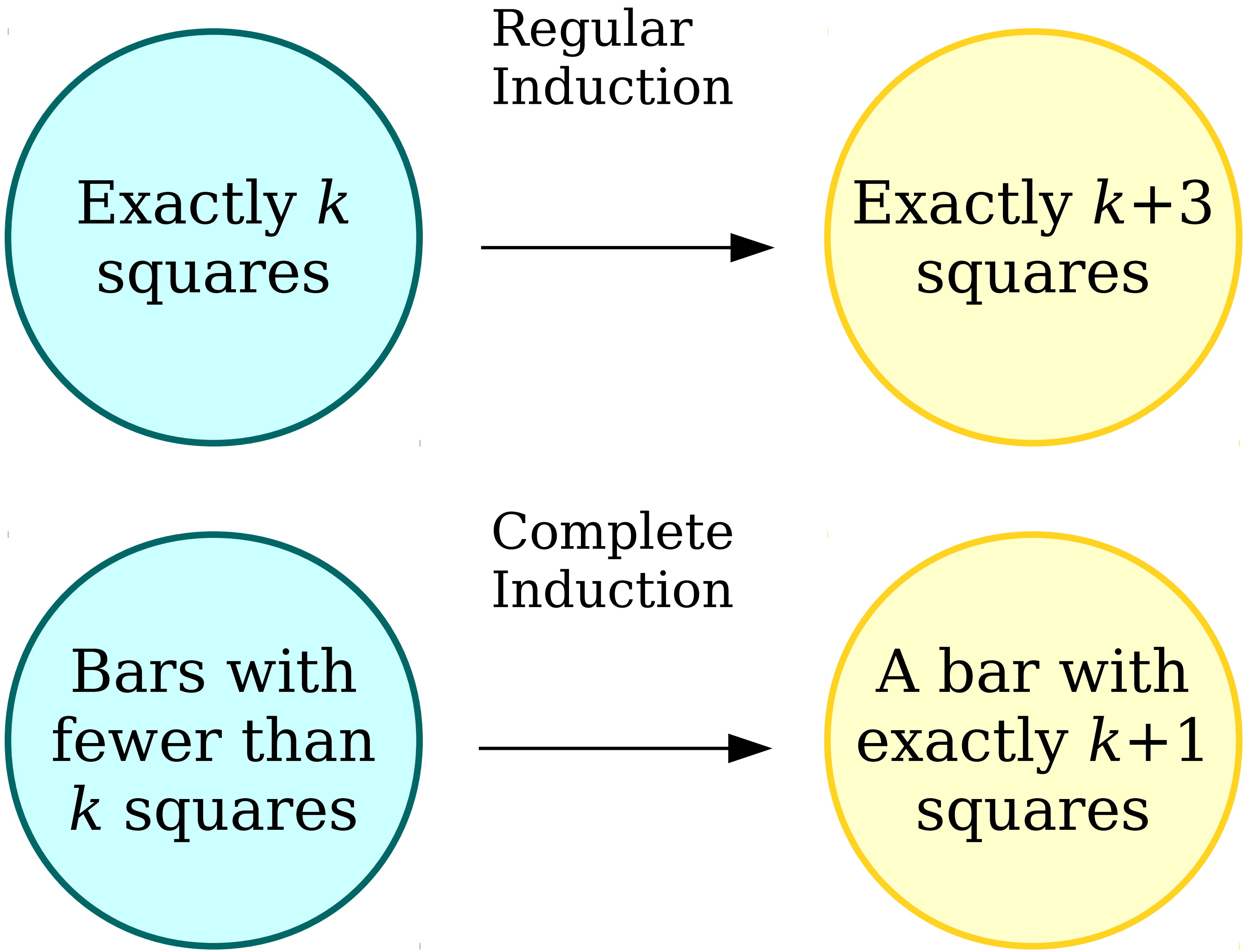
Induction vs. Complete Induction



Regular Induction

Complete Induction

Induction vs. Complete Induction



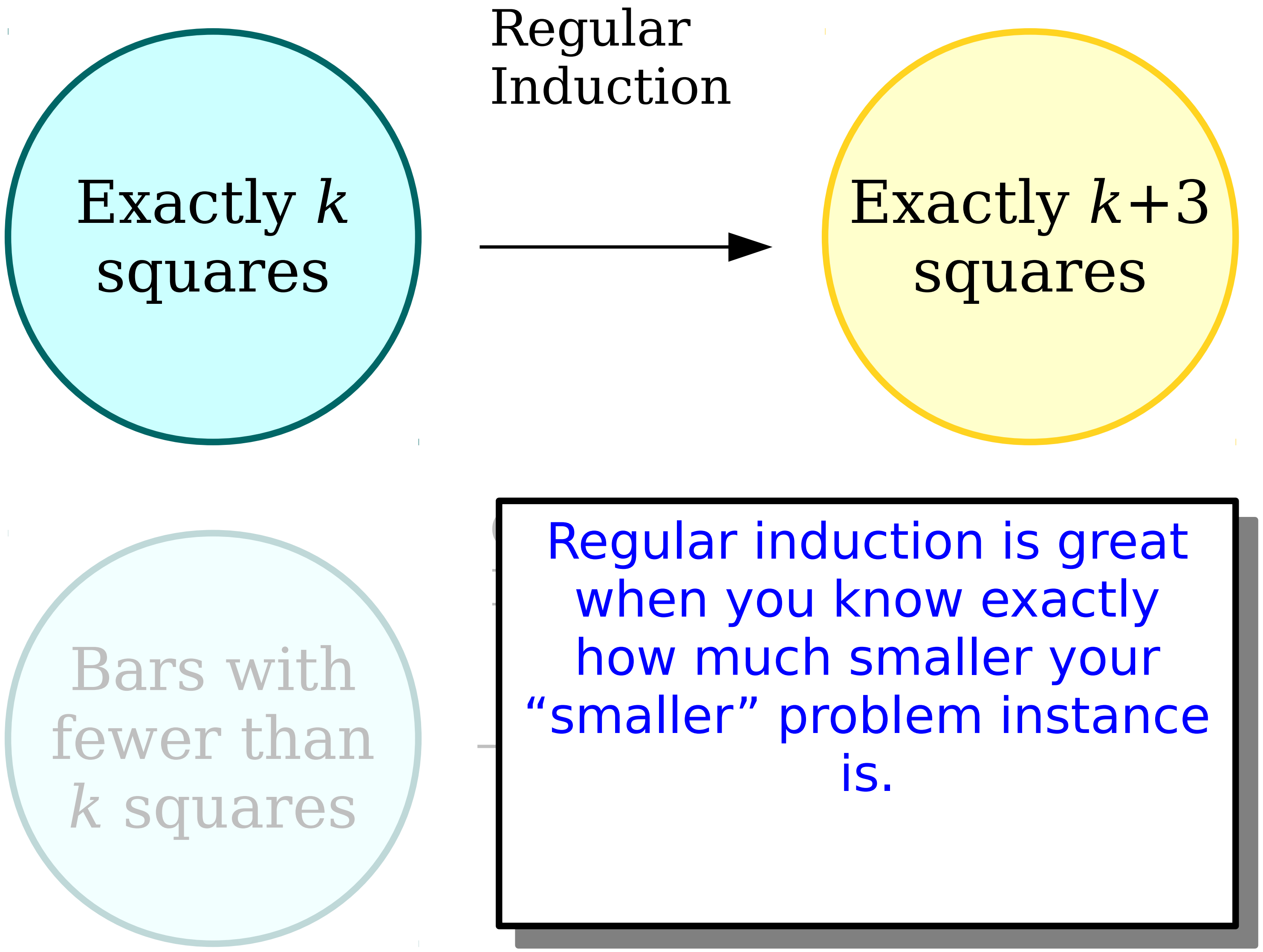
Regular Induction

Exactly k squares → Exactly $k+3$ squares

Complete Induction

Bars with fewer than k squares → A bar with exactly $k+1$ squares

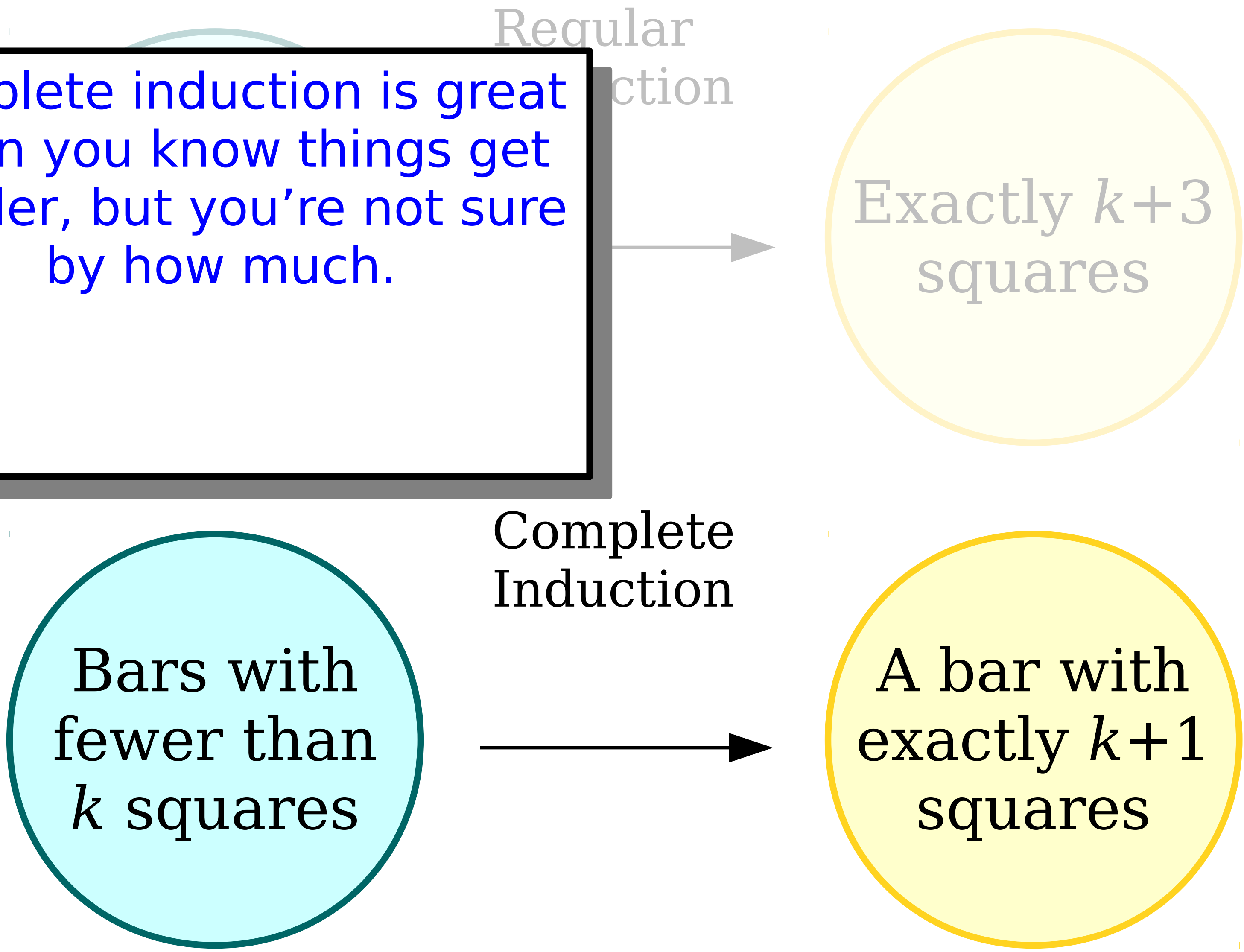
Induction vs. Complete Induction



Regular induction is great when you know exactly how much smaller your “smaller” problem instance is.

Induction vs. Complete Induction

Complete induction is great when you know things get smaller, but you're not sure by how much.



Regular
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Exactly $k+3$ squares

Complete
Induction

Bars with fewer than k squares

A bar with exactly $k+1$ squares

An Important Milestone

Recap: ***Discrete Mathematics***

- The past five weeks have focused exclusively on discrete mathematics:
    - Induction
    - Graphs
    - Formal Proofs
    - Set Theory
    - Functions
    - The Pigeonhole Principle
    - Mathematical Logic
    - Cardinality
- These are building blocks we will use throughout the rest of the quarter.
- These are building blocks you will use throughout the rest of your CS career.

Next Up: ***Computability Theory***

- It's time to switch gears and address the limits of what can be computed.
- We'll explore these questions:
  - How do we model computation itself?
  - What exactly is a computing device?
  - What problems can be solved by computers?
  - What problems *can*'*t* be solved by computers?
- ***Get ready to explore the boundaries of what computers could ever be made to do.***

Next Time

- ***Formal Language Theory***
  - How are we going to formally model computation?
- ***Finite Automata***
  - A simple but powerful computing device made entirely of math!
- ***DFAs***
  - A fundamental building block in computing.

Ramsey Revisited

Ramsey Revisited

- In lecture, we proved the Theorem on Friends and Strangers: any 6-clique whose edges are painted one of two colors contains a monochrome triangle.
- On PS4, you proved that any 17-clique whose edges are painted one of three colors has a monochrome triangle.
- What about if you use four colors? Five colors? Six colors?

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The notation $n!$ represents ***n factorial***, the product of all natural numbers between 1 and n, inclusive.

$5! = 1 \times 2 \times 3 \times 4 \times 5.$

The value $3n!$ is read as $3(n!)$.

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Next, pick a natural number $k \geq 1$ and assume $P(k)$ is true, that any coloring of the edges of a $3k!$-clique with k colors has a monochrome triangle. We need to show $P(k+1)$ is true. To do so, pick a coloring of the edges of a $3(k+1)!$-clique with $k+1$ colors. We need to find a monochrome triangle.

Pick any node v in the clique and look at the edges incident to v. There are $3(k+1)! - 1$ other nodes in the clique and $k+1$ colors. By the generalized pigeonhole principle, this means v is adjacent to at least

$$\left\lceil \frac{3(k+1)! - 1}{k+1} \right\rceil = \left\lceil 3k! - \frac{1}{k+1} \right\rceil = 3k!$$

nodes by edges of the same color. Assume WLOG that color is blue.

Theorem: If $n \geq 1$ is a natural number, then for any way of painting the edges of a $3n!$-clique with n colors, the clique has a monochrome triangle.

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More on Ramsey Triangles

- We've proved that $3n!$ nodes is enough to get a triangle with $n \geq 1$ colors on the edges.
- For $n = 3$, this says we need 18 nodes, but as you proved on PS4 you can do this with 17 nodes.
- For $n = 4$, this says we need 72 nodes. We know that 50 nodes is too few and 66 nodes is enough. The actual answer is therefore somewhere between 51 and 66.
- ***Open problem:*** Find the exact minimum number of nodes needed to get a monochrome triangle with $n \geq 4$ colors.
- ***Challenge problem:*** Show that $\lceil e \cdot n! \rceil$ nodes is always sufficient to get a monochrome triangle with $n \geq 1$ colors. *(This is hard but doable if you know the material from CS103, plus the Taylor series for e. Come talk to me if you want more details.)*